SECTION B8.4

TORSION OF THIN-WALLED OPEN SECTIONS

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B8.4.0 TORSION OF THIN-WALLED OPEN SECTIONS

An open section is a section in which the centerline of the wall does not form a closed curve. Channels, angles, I-beams, and wide-flange sections are among many common structural shapes characterized by combinations of thin-walled rectangular elements; a variety of thin-walled curved sections is used in aircraft and missile structures. The basic characteristic of these sections is that the thickness of the component element is small in comparison with the other dimensions.

The torsional analysis of thin-walled open sections for both unrestrained and restrained torsion is included in this section. Torsional shear stress, angle of twist, and warping deformations are determined for unrestrained torsion. Torsional shear stress, warping shear stress, warping normal stress, angle of twist, and the first, second, and third derivatives of angle of twist are determined for restrained torsion.

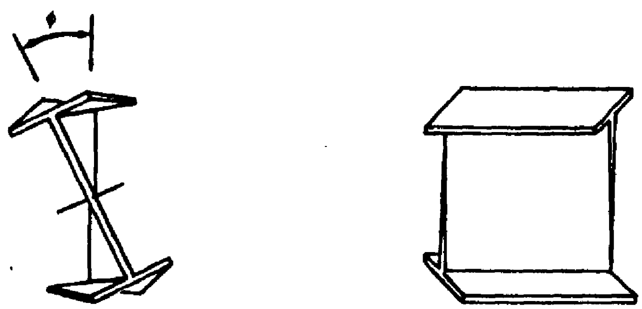
B8.4.1 GENERAL

The stresses and deformations determined by the equations in this section can be superimposed with bending and axial load stress and deformations if the limitations of Section B8.4.1-II are not exceeded and proper consideration of stress and deformation sign convention is taken into account.

B8.4.1 GENERAL

I. BASIC THEORY

If a member of open cross section is twisted by couples applied at the ends in the plane perpendicular to the axis of the bar and the ends are free to warp, we have the case of unrestrained torsion (Fig. B8.4.1-1).



A. Rotated Section B. Warped Section

Figure B8.4.1-1. Warping

However, if cross sections are not free to warp or if the torque varies along the length of the bar, warping varies along the bar and torsion is accompanied by tension or compression of longitudinal fibers. Also, the rate of change of the angle of twist along the bar's longitudinal axis varies. This case is called restrained torsion.

These two types of torsion will be discussed separately in the following sections.

A. Unrestrained Torsion

The twisting moment on thin-walled open sections is resisted only by the torsional shear stress for unrestrained torsion. However, the manner in which a thin-walled open section carries a torsional moment differs from the manner in which the thin-walled closed section carries a torsional moment. This difference can be seen by comparison of Figures B8.2.1-1A, B8.2.2-2A, and B8.3.1-1A to Figure 8.4.1-2. The thin-walled closed section carries the load by a shear flow that goes around the section, while the open section carries the load by a shear flow which goes around the perimeter of the section. From Figure B8.4.1-2, it can be seen that the shear stress distribution across the thickness of the section is linear and that the maximum stress on one edge is equal to the negative of the maximum stress on the other edge. (Ref. 1).

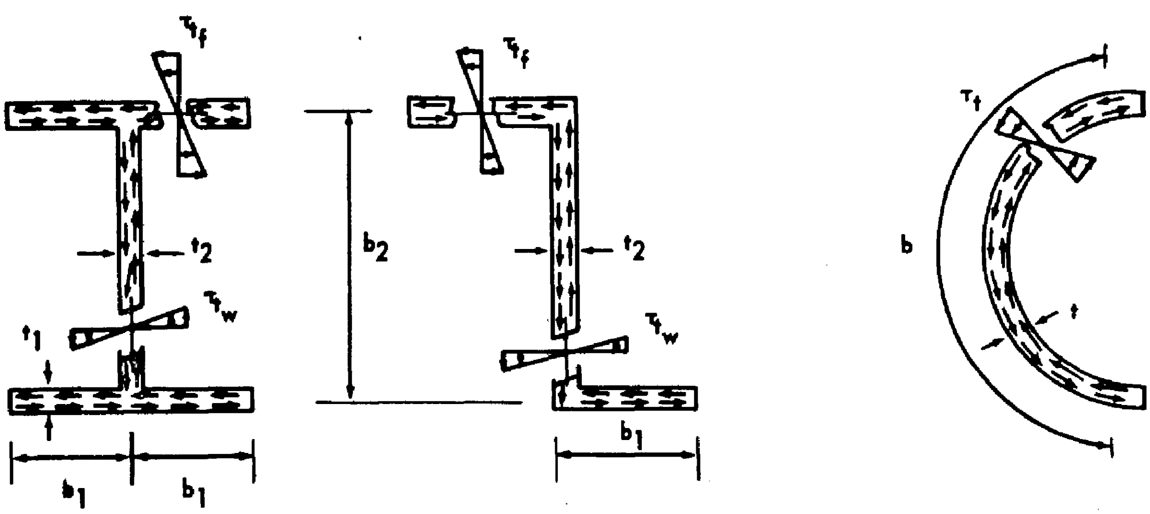


Figure B8.4.1-2. Pure Torsion Shear Stress Distribution

The torsional analysis of thin-walled open sections for unrestrained torsion will require that the torsional shear stress (τ_t) be determined at any point (P) on the section. Because of the definition of unrestrained torsion, the torsional shear stress at any point on the section will remain constant throughout the length of the member.

The angle of twist of the cross section (ϕ) should also be determined, plus the warping deformation (w) at any point (P) on the cross section.

As was the case for solid sections and thin-walled closed sections, two unique coefficients exist that characterize the geometry of each cross section. These coefficients are called the torsional constant (K) and the torsional modulus (S_t) and are functions of the dimensions of the cross section. These coefficients are discussed in detail below (Section B8.4.1-IV).

B. Restrained Torsion

When a member with a thin-walled open cross section is restrained against warping a complex distribution of longitudinal stresses is developed that cannot be evaluated using elementary theories. The assumption that plane sections remain plane during deformation is no longer valid, and applications of Saint-Venant's principle may lead to serious errors. In thin-walled open sections, stresses produced by restrained warping diminish very slowly from their points of application and may constitute the primary stress system developed in the member.

Obviously, if one section is restrained in such a way that it cannot warp, a system of normal stresses must be developed to eliminate this warping. In general, these normal stresses vary from point to point along the member and, hence, they are accompanied by a nonuniform shearing stress distribution. This, in turn, alters the twist of the section. As a result, the twisting moment developed on each section is no longer proportional to the rate of twist, and final shearing stresses cannot be obtained by those that were produced by unrestrained torsion.

Therefore, three types of stresses must be evaluated for the case of restrained torsion. These are: (1) pure torsional shear stress, (2) warping shear stress, and (3) warping normal stress. These stress distributions are shown for several common sections in Figures B8.4.1-3, B8.4.1-4, and B8.4.1-5. It will be required to evaluate these stresses at any point (P) on the cross section and at any arbitrary distance (L_x) from the origin. Also, the angle of twist (ϕ) should be determined between an arbitrary cross section and the origin along with the warping deformation (w) at any point (P) on an arbitrary cross section. **(Ref. 2).**

It was shown previously that two coefficients were necessary to characterize the geometry of the cross section for unrestrained torsion. These were the torsional constant (K) and the torsional section modulus (S_t). For restrained torsion, three additional coefficients are required to characterize fully the geometry of the cross section and the point where the stresses are to be determined. These coefficients are called the warping constant (Γ) — a function of the dimensions of the cross section, the normalized warping function (W_n), and warping statical moment (S_w). The latter two are functions of both the dimensions of the cross section and a specific point on the cross section. These coefficients are discussed in detail in Section B8.4.1-IV.

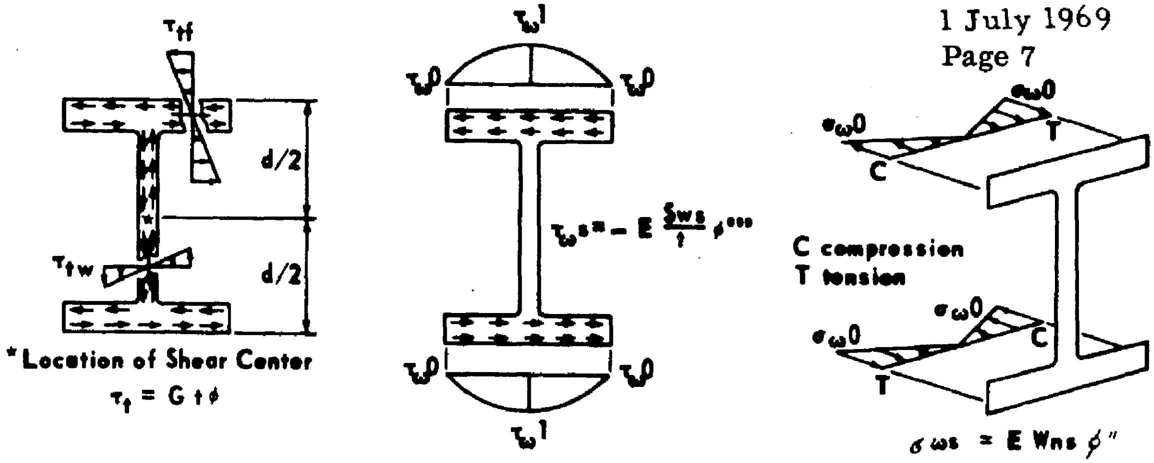


Figure B8.4.1-3. Restrained Warping Stress in I-Section

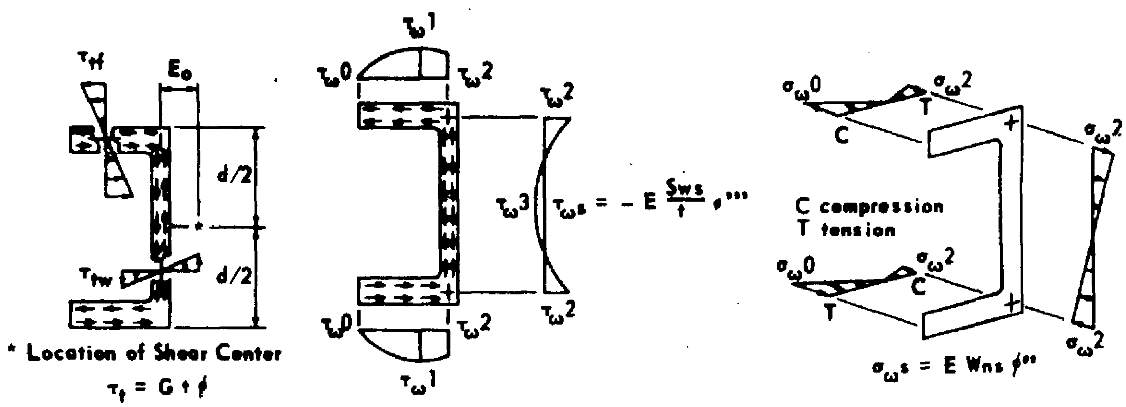


Figure B8.4.1-4. Restrained Warping Stresses in Channel Section

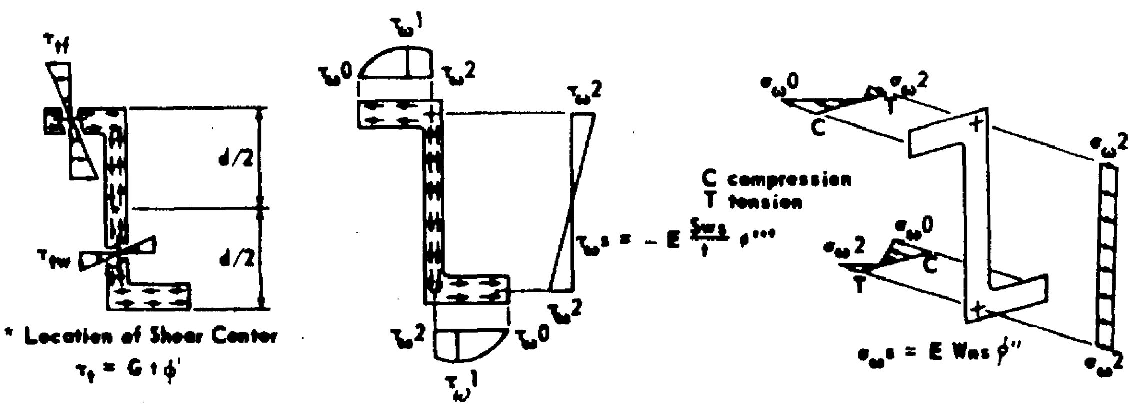


Figure B8.4.1-5. Restrained Warping Stresses in Z-Section

B8.4.1 GENERAL

II. LIMITATIONS

The torsional analysis of thin-walled open sections is subject to the following limitations:

A. Homogeneous and isotropic material

B. Thin-walled cross section not necessarily of constant thickness

C. No abrupt variations in thickness except at reentrant corners

D. No buckling

E. Inexact calculations of stresses at points of constraint and at abrupt changes of applied twisting moment

F. Applied twisting moment cannot be impact load

G. No abrupt changes can occur in cross section

H. Shear stress is within shearing proportional limit and proportional to the shear strain (elastic analysis).

I. Points of constraint are fully fixed, and no partial fixity is allowed.

B8.4.1 GENERAL

III. MEMBRANE ANALOGY

In the case of a narrow rectangular cross section, the membrane analogy gives a very simple solution to the torsional problem. Neglecting the effect of the short sides of the rectangle and assuming that the surface of the slightly deflected membrane is cylindrical (Figure B8.4.1-6), the deflection is

$$H = \frac{pt^2}{8T}$$

and the maximum slope is $\frac{pt}{2T}$. The volume bounded by the deflected membrane and the xy plane is (Ref. 3):

$$V = \frac{pbt^3}{12T} .$$

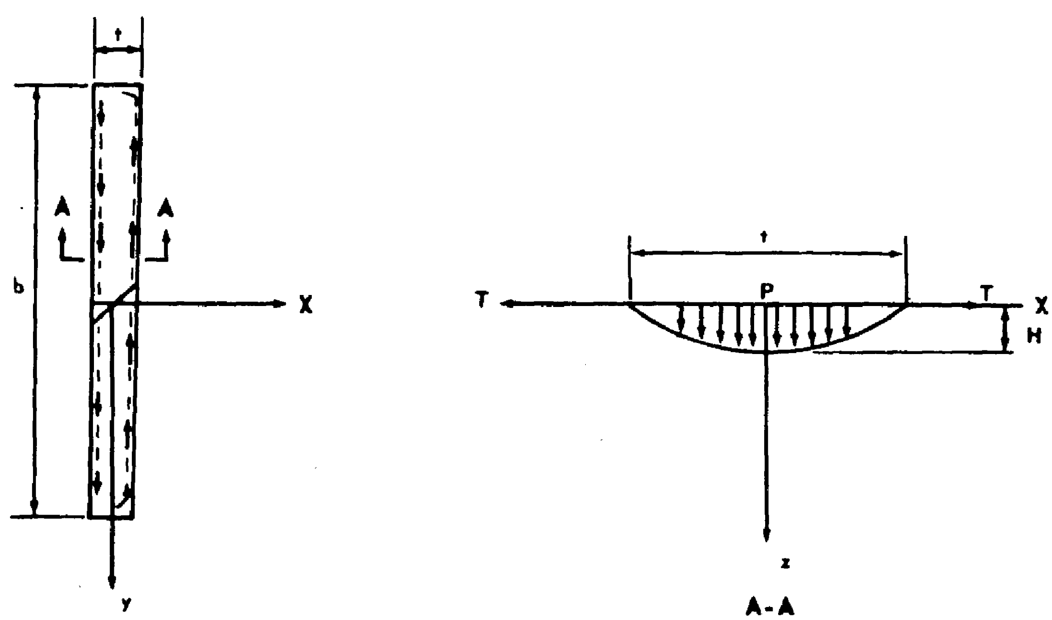


Figure B8.4.1-6. Membrane Analogy for Torsion of Thin Rectangular Section

Now using membrane analogy and substituting $2G\theta$ for p/T in the previous equations, the twisting moment (M_t) is given by

$$M_t = \frac{1}{3} bt^3 \ G\theta$$

or

$$\theta = M_t / 1/3bt^3G = \frac{M_t}{S_t G t}$$

and the maximum shearing stress is

$$\tau_{max} = tG\theta = \frac{M_t}{\frac{1}{3} bt^2} = \frac{M_t}{S_t}$$

where

$$S_t = 1/3bt^2 \ .$$

The equations for M_t and τ_{max}, obtained for a thin rectangle can also be used for cross sections, such as those shown in Figure B8.4.1-2, by simply adding the expression $1/3bt^3$ for each element of the section (neglecting a small error at corners or points of intersection of the elements). In the general case of a section with N elements:

$$K = 1/3 \sum_{i=1}^{n} b_i t_i^3 \ .$$

The maximum shearing stress on any element i is given by

$$(\tau_{max})_i = \frac{M_t (t_i)_{max}}{S_t} \quad .$$

The maximum stress on the entire section is given by

$$\tau_{max} = \frac{M_t t_{max}}{S_t} \quad .$$

B8.4.1 GENERAL

IV. TORSIONAL COEFFICIENTS

In the development of the formulas for the torsional analysis of open cross sections, it is convenient to designate certain terms as torsional coefficients for the cross section. The terms K and Γ are properties of the entire cross section, while the terms S_t, S_w, and W_n apply to specific points on the cross section.

A. Torsional Constant (K)

The torsional coefficient (K) is called the torsional constant, and its value depends upon the geometry of the cross section.

Torsional constants for thin-walled open sections are based on formulas for the thin rectangle.

Section B8.2.2-III contains an expression for the torsional constant for a general rectangular section. Since we are concerned with a thin-walled section, it can be seen that when the length-to-thickness ratio is approximately ten, the value of the torsion constant is

$$K \cong 1/3bt^3 .$$

This value is also verified by the membrane analogy in Section B8.4.1-III. The torsional constant for curved elements is the same as that for a rectangle with b denoting the length of the curved element, as shown in Figure B8.4.1-2.

Therefore, for a section composed of many thin rectangular, or thin curved elements, the torsional constant can be evaluated by the following expression:

$$K = 1/3 \sum_{i}^{n} b_i t_i^3 \quad .$$

If a section has any element with a length-to-thickness ratio less than ten, the value of K for that element should be determined by the equations in Section B8.2.2-III.

More accurate torsional constant expressions are determined for some standard sections by considering the junctions of the rectangles and rounded fillets at the junctions.

Some K values for frequently used sections are (Ref. 2):

1. For I sections with uniform flanges (Fig. B8.4.1-7A):

$$K = 2/3 b t_f^3 + 1/3(d-2t_f) t_w^3 + 2\alpha D^4 - 0.42016 t_f^4$$

where

$$\alpha = 0.094 + 0.07 \frac{R}{t_f}$$

$$D = \frac{(t_f + R)^2 + t_w \left(R + \frac{t_w}{4}\right)}{2R + t_f}$$

2. For I sections with sloping flanges (Fig. B8.4.1-7B):

$$K = \frac{b - t_w}{6} (t_f + a)(t_f^2 + a^2) + 2/3\, t_w a^3 + 1/3(d - 2a) t_w^3$$

$$+ 2\alpha D^4 - E t_f^4$$

where

$$D = \frac{(F + c)^2 + t_w \left(R + \frac{t_w}{4}\right)}{F + R + c}$$

and for 5-percent flange slope

$$\alpha = 0.066 + 0.021 \frac{t_w}{a} + 0.072 \frac{R}{a}$$

$$E = 0.44104$$

$$F = \frac{R}{20} \left(19.0250 - \frac{t_w}{2R}\right)$$

and for 2-percent flange slope

$$\alpha = 0.084 + 0.007 \frac{t_w}{a} + 0.071 \frac{R}{a}$$

$$E = 0.42828$$

$$F = \frac{R}{50} \left(49.01 - \frac{t_w}{2R}\right) \quad .$$

3. For channels with sloping flanges (Fig. B8.4.1-7C):

$$K = 1/3\, t_w^3 d + \frac{b'}{6} (a + t_f) (a^2 + t_f^2) \quad .$$

4. For Tee section (Fig. B8.4.1-7D):

$$K = \frac{bt_f^3}{3} + \frac{ht_w^3}{3} + \alpha\ D^4$$

where

$$\alpha = 0.094 + 0.07\ R/t_f$$

$$D = \frac{(t + R)^2 + t_w \left(R + \frac{t_w}{4}\right)}{2R + t_f}$$

5. Angle section (Fig. B8.4.1-7E):

$$K = 1/3bt_1^3 + 1/3\ dt_2^3 + \alpha\ D^4$$

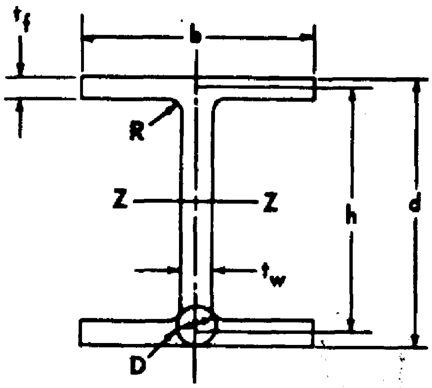
where

$$\alpha = \frac{t_2}{t_1}\left(0.07 + 0.076\ \frac{R}{t_1}\right) \qquad t_1 \gtreqless t_2$$

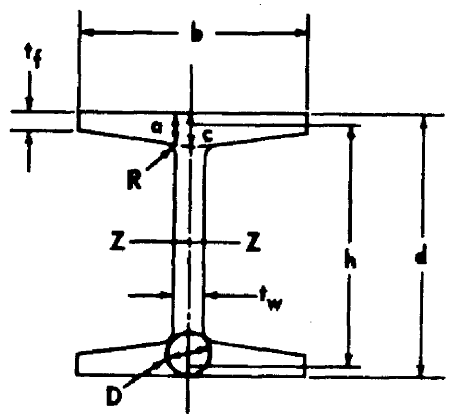
$$D \cong \frac{(t_1 + R)^2 + t_2 \left(R + \frac{t_2}{4}\right)}{2R + t_1}$$

6. Zee section and channel section with uniform flanges (Fig. B8.4.1-7F).

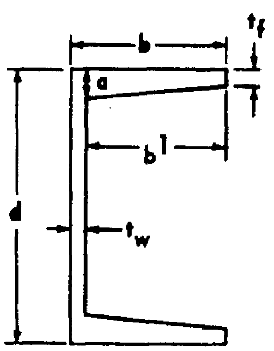
K values for these sections can be calculated by summing the K's of the constituent angle sections computed in case 5.



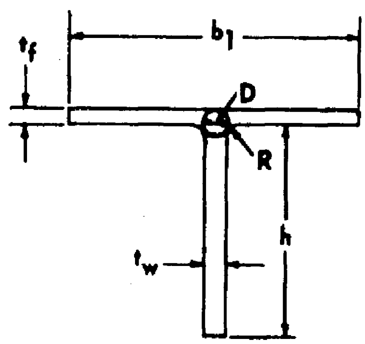
A. Uniform Flanges



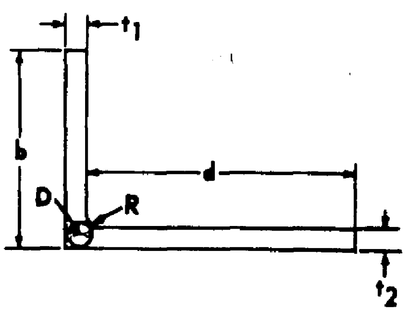
B. Sloping Flanges



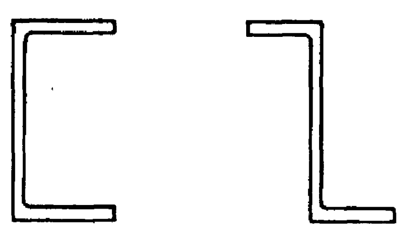
C. Channel with Sloping Flanges



D. Tee Section



E. Angle



F. Zee and Uniform Channel

Figure B8.4.1-7. Frequently Used Sections

It should be noted that the K formulas for these frequently used sections are based on membrane analogy and on reasonably close approximations giving results that are rarely as much as 10 percent in error.

B. Torsional Modulus (S_t)

The torsional coefficient (S_t) is called the torsional modulus. Its value for any point on the section depends upon the geometry of the cross section.

The basic equation for determining the torsional modulus at an arbitrary point (s) on a cross section is:

$$S_t(s) = \frac{K}{t(s)}$$

where K is as defined in Section B8.4.1-IVA and t(s) is the thickness of the section at point (s).

Because the torsional modulus is necessary for the calculation of the torsional shear stress in the equation

$$t = \frac{M\ (x)}{S_t(x, s)}$$

it is often required to find the minimum value of $S_t(s)$ in order to make τ_t a maximum.

Therefore:

$$S_t(\min) = \frac{K}{t_{max}}$$

where t_{max} is the maximum thickness in the cross section.

C. Normalized Warping Function (W_n)

The torsional coefficient (W_n) is called the normalized warping function. Its value depends upon the geometry of the cross section and upon specific points on the cross section.

For the generalized section shown in Figure B8.4.1-8, the following equation is used for calculating $W_n(s)$ at any point (s) on the section:

$$W_n(s) = \frac{1}{A} \int_0^b W_{os} \, tds - w_{os}$$

where

$$A = \int_0^b tds$$

$$W_{os} = \int_0^s \rho_o \, ds.$$

Some W_n values for frequently used sections include:

1. For symmetrical wide flange and I-shapes (Fig. B8.4.1-9A):

$$W_{no} = \frac{bh}{4} \quad .$$

2. For channel sections (Fig. B8.4.1-9B):

$$W_{no} = \frac{uh}{2}$$

$$W_{n2} = \frac{E_o h}{2}$$

ρ Perpendicular distance to tangent line from centroid
ρ_o Perpendicular distance to tangent line from shear center
cg Centroid of cross section
sc Shear center of cross section
z,y Coordinates referred to the principal centroidal axes
φ Angle of twist

[All directions are shown positive. ρ and ρ_o are positive if they are on the left side of an observer at P (z,y) facing the positive direction of s.]

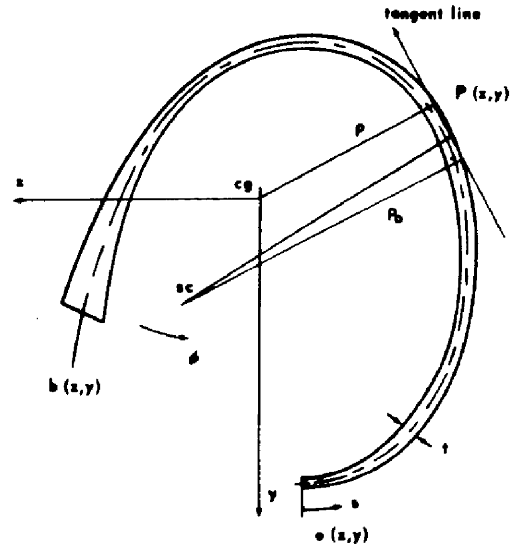


Figure B8.4.1-8. General Thin-Walled Open Cross Section

where

$$E_o = \frac{(b')^2 t}{2b't + h\, t_w/3}$$

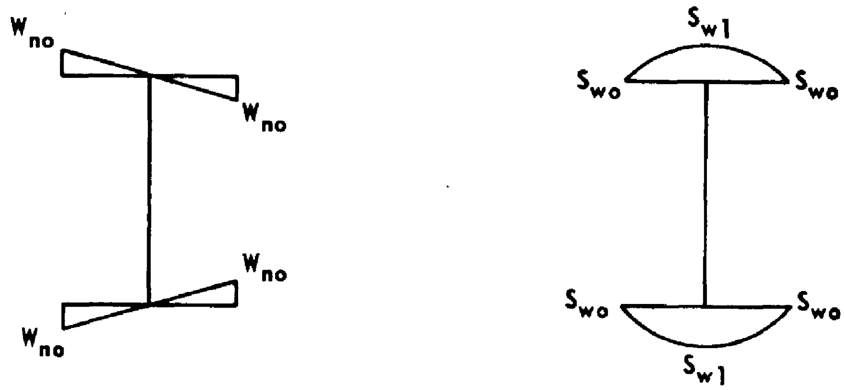
and

$$u = b' - E_o \; .$$

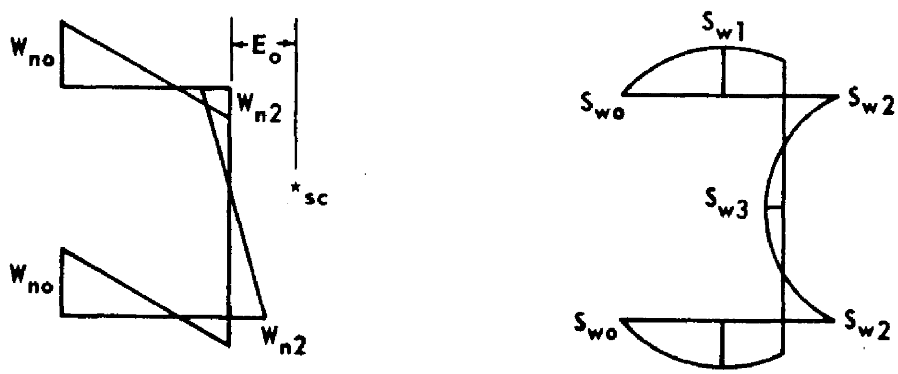
3. For zee sections (Fig. B8.4.1-9C):

$$W_{no} = \frac{uh}{2}$$

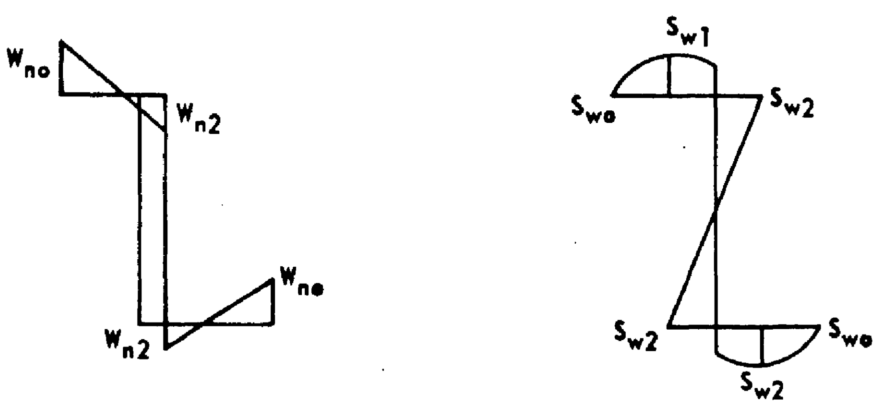
$$W_{n2} = \frac{u'h}{2}$$



A. Symmetrical H-and I-Sections



B. Channel Sections



C. Zee Sections

Figure B8.4.1-9. Distribution of W_n and S_w for Standard Sections

where

$$u = b' - u'$$

$$u' = \frac{(b')^2 t}{h_{tw} + 2b't} \quad .$$

In the foregoing expressions:

h = distance between centerlines of flanges, in.

b = flange width, in.

b' = distance between toe of flange and centerline of web, in.

t = average thickness of flange, in.

t_w = thickness of web, in.

D. <u>Warping Statical Moments (S_w)</u>

The torsional coefficient (S_w) is called the warping statical moment. Its value depends upon the geometry of the cross section and upon specific points on the cross section.

For the generalized section shown in Figure B8.4.1-8, the following equation is used for calculating $S_w(s)$ at any point (s) on the section:

$$S_w(s) = \int_0^s W_n(s)\, t\, ds.$$

The value of $W_n(s)$ is determined from the previous subsection (B8.4.1-IVC). Some S_w values for frequently used sections include:

1. For symmetrical wide flange and I-shapes (Figure B8.4.1-9A):

$$S_{w1} = \frac{hb^2t}{16} \quad .$$

2. For channel sections (Figure B8.4.1-9B):

$$S_{w1} = \frac{u^2ht}{4}$$

$$S_{w2} = \frac{(b' - 2E_o)hb't}{4}$$

$$S_{w3} = \frac{(b' - 2E_o)hb't}{4} - \frac{E_o h^2 t_w}{8} \quad .$$

3. For zee sections (Fig. B8.4.1-9C):

$$S_{w1} = \frac{(ht_w + b't)^2 \, h(b')^2 t}{4(ht_a + 2b't)^2}$$

$$S_{w2} = \frac{h^2 t_w (b')^2 t}{4(ht_w + 2b't)}$$

where u, h, t, b, b', E_o, and t_w are defined in the previous section.

E. Warping Constant (Γ)

The torsional coefficient (Γ) is called the warping constant. Its value depends only on the geometry of the cross section. For the generalized section shown in Figure B8.4.1-8, the following equation is used for calculating Γ:

$$\Gamma = \int_{o}^{b} W_n(s)\, t^2 ds \; .$$

The value of $W_n(s)$ is determined from Section B8.4.1-IVC. Some values for frequently used sections are:

1. For symmetrical wide flange and I-shapes (Fig. B8.4.1-9A):

$$\Gamma = \frac{h^2 b^3 t}{24} = \frac{I_y h^2}{4} \; .$$

2. For channel sections (Fig. B8.4.1-9B):

$$\Gamma = 1/6\ (b'3E_o)h^2(b')\ t + F_o^2 I_x \; .$$

3. For zee sections:

$$\Gamma = \frac{(b')^3 th^2}{12} \; \frac{b't + 2ht_w}{nt_w + 2b't}$$

where h, b, t, t_w, b', and E_o are defined in Section B8.4.1-IVC, I_x = the moment of inertia of the entire section about the xx axis, and I_y = the moment of inertia of the entire section about the yy axis.

B8.4.2 UNRESTRAINED TORSION

The formulas given in this section apply only to members of open cross section twisted by couples applied at the ends in the plane perpendicular to the longitudinal axis of the bar, and the ends are free to warp as shown in Figure B8.4.1-1.

B8.4.2 UNRESTRAINED TORSION

I. ANGLE OF TWIST

For the case of unrestrained torsion, the torsional moment resisted by the cross section is

$$M_1 = GK\,\phi'$$

where

M_1 = resisting moment of unrestrained cross section, in.-lb
$= M_t$

G = shear modulus of elasticity, psi

K = torsional constant for the cross section, $in.^4$

$\phi' = \frac{d\phi}{dX}$ = angle of twist per unit of length.

This is the first derivative of the angle of rotation ϕ with respect to X, the distance measured along the length of the member from the left (Fig. B8.4.2-1).

Therefore, the basic equation for determining the angle of twist between the origin and an arbitrary cross section at a distance L_x from the origin is:

$$\phi(x) = \frac{1}{G} \int_0^{L_x} \frac{M(x)}{K(x)}\, dx$$

where K(x) is the torsion constant at L_x. If the cross section does not vary and M(x) is taken as M_t applied at the end of the member, the angle of twist is determined from the equation:

$$\phi(x) = \frac{M_t L_x}{GK} .$$

The total twist of the bar is:

$$\phi(\max) = \frac{M_t L}{GK} .$$

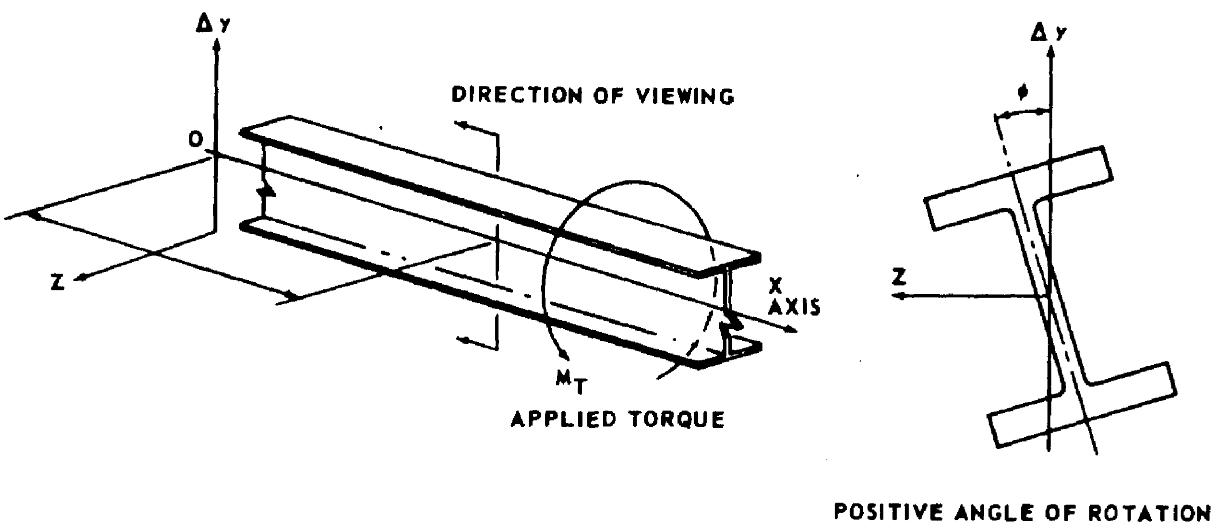


Figure B8.4.2-1. General Orientation

B8.4.2 UNRESTRAINED TORSION

II. STRESSES

The twisting moment (M_t) on thin-walled open sections is resisted only by the torsional shear stress for unrestrained torsion. The torsional shear stress at the edge of an element is determined by the formula:

$$\tau_t = Gt\,\phi' \;.$$

Because $\phi = \frac{M(x)}{GK}$ the basic equation for determining the torsional shear stress at an arbitrary point (s) on an arbitrary cross section is:

$$\tau_t = \frac{M(x)}{S_t(x, s)}$$

where

$$S_t(x, s) = \frac{K(x)}{t(x, s)} \quad .$$

K(x) is evaluated at $x = L_x$ where the torsional shear stress is to be determined, and t(x, s) is evaluated at the arbitrary cross section and at the point(s) on the arbitrary cross section.

If the member has uniform cross section and M(x) is taken as M_t, applied to the end of the bar, the equation reduces to:

$$\tau_t = \frac{M_t}{S_t(s)}$$

where

$$S_t(s) = \frac{K}{t(s)} .$$

The maximum stress τ_t will occur on the thickest element; $t(s)$ is maximum.

B8.4.2 UNRESTRAINED TORSION

III. WARPING DEFORMATION

The basic equation for determining the warping deformation w(s) at any point on an arbitrary cross section at a distance $x = L_x$ from the origin is

$$w(s) - w_o = \phi' \int_0^s r\,ds$$

where w(s) is the warping deformation at point (s) on the middle line of the cross section in the x direction; w_o is the displacement in the x-direction of the point from which s is measured; r(s) is the distance of the tangent of arc length ds from point o, taken positive if a vector along the tangent and pointing in the direction of increasing s gives a positive moment with respect to the axis of rotation (Fig. B8.4.2-2); ϕ' is determined from Section B8.4.2-I.

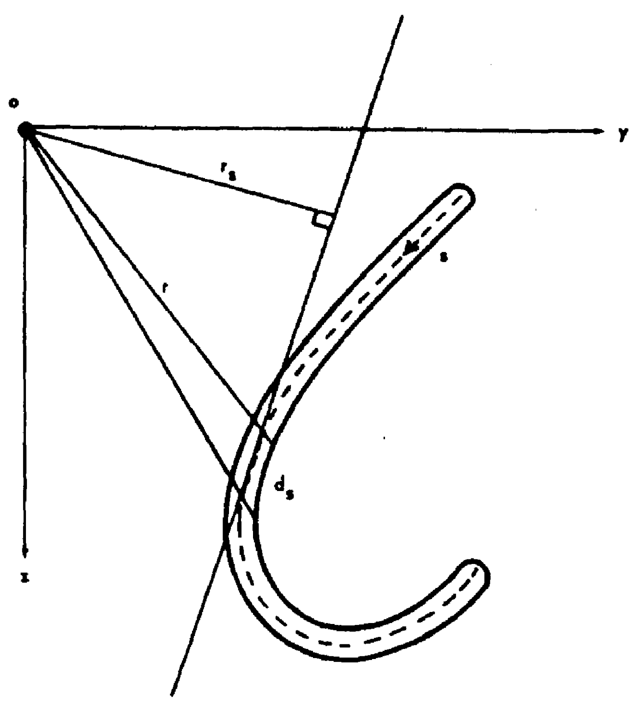


Figure B8.4.2-2. General Section

For the case of unrestrained torsion, the point 0 can be located arbitrarily.

The warping of the cross section with respect to the plane of average w has been found to be

$$w(s) = \phi' W_n(s)$$

where $W_n(s)$ is the normalized warping function found in Section B8.4.1-IVC.

B8.4.2 UNRESTRAINED TORSION

IV. STRESS CONCENTRATION FACTORS

Stress concentrations occur in composite cross sections at any reentrant corner; that is, at the intersection of the web and either of the flanges in the I-section or at the interior angle joining the two legs of the angle section. Exact analysis of stress concentrations at these points is very difficult and must be carried out experimentally, usually by membrane analogy.

For many common sections, the maximum stress at the concave or reentrant point is

$$\tau_{max} = K_3 \, G \, \phi'$$

where (Ref. 4)

$$K_3 = \frac{D}{1 + \frac{\pi^2 D^4}{16A^2}} \left\{ 1 + \left[0.118 \ln \left(1 + \frac{D}{2\rho} \right) + 0.238 \frac{D}{2\rho} \right] \tanh \frac{2\theta}{\pi} \right\}$$

D = diameter of largest inscribed circle (Section B8.4.1-IVA)

A = cross-sectional area

ρ = radius of concave boundary at the point (positive)

θ = angle through which a tangent to the boundary rotates in rolling around the concave portion, rad.

For angles with legs of equal thickness, the percentage increase of stress in fillets is shown on Figure B8.4.2-3.

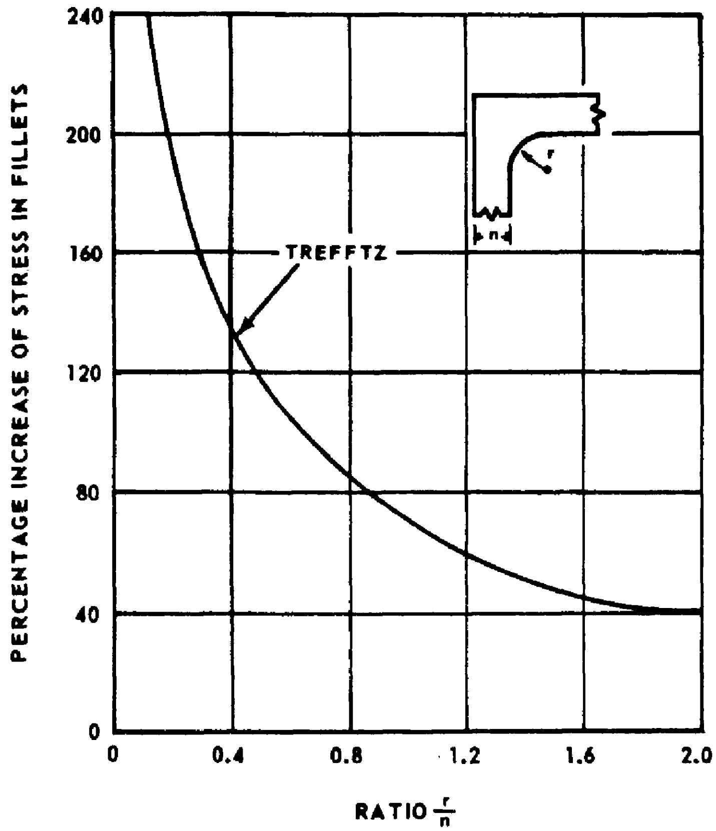


Figure B8.4.2-3. Stress Increase in Fillets of Angles

B8.4.4 EXAMPLE PROBLEMS

I. UNRESTRAINED TORSION

A member with an unsymmetrical section shown in Figure B8.4.4-1 is loaded by an end moment and is free to warp. If M_t = 100 in.-lbs and L = 41 in., determine the maximum angle of twist, torsional shear stress, and warping deformations.

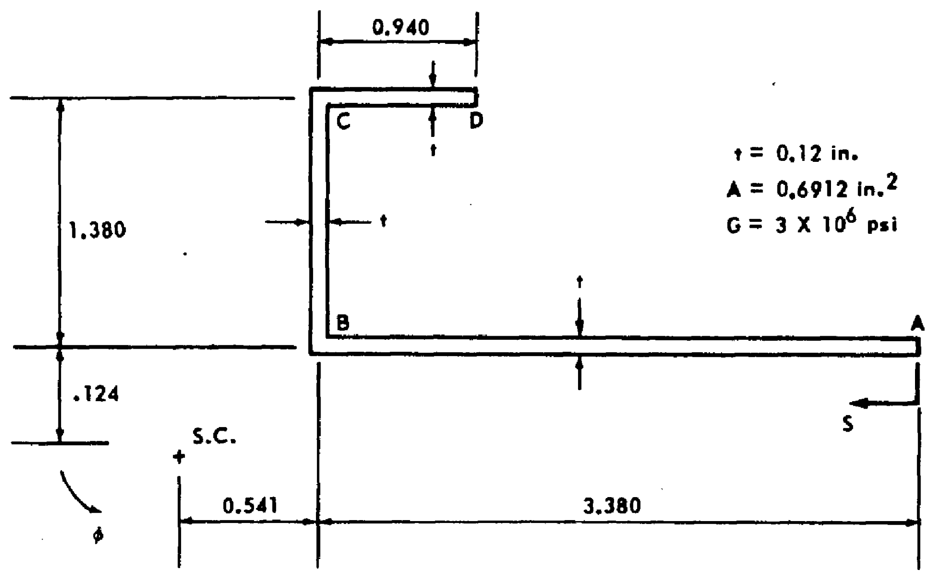


Figure B8.4.4-1. Cross Section for Example Problem I

Evaluate $W_{os} = \int_0^s \rho_o \, ds$:

$W_{os} = 0.124s$ for $s < 3.380$

$W_{os} = 0.419 + 0.541s$ for $4.760 > s > 3.380$

$W_{os} = 1.166 - 1.504s$ for $s > 4.760$

Evaluate $W_n(s)$:

$$W_n(s) = \frac{1}{A}\int_o^b W_{os}\,t\,ds - W_{os}$$

$$\frac{1}{A}\int_o^b W_{os}\,t\,ds = \frac{0.12}{0.6912}\left\{\int_o^{3.38} 0.124s\,ds + \int_o^{1.38}(0.419 + 0.541s)\,ds + \int_o^{0.94}(1.166 - 1.504s)\,ds\right\} = 0.388.$$

Then:

$$W_n(s) = 0.388 - 0.124s \qquad \text{for } s < 3.380$$

$$W_n(s) = -0.031 - 0.541s \qquad \text{for } 4.760 > s > 3.380$$

$$W_n(s) = -0.778 + 1.504s \qquad \text{for } s > 4.76 .$$

Therefore, at points on the cross section:

$$W_n(A) = 0.388$$

$$W_n(B) = -0.031$$

$$W_n(C) = -0.778$$

$$W_n(D) = 0.636 .$$

The distribution of $W_n(s)$ is shown in Figure B8.4.4-2.

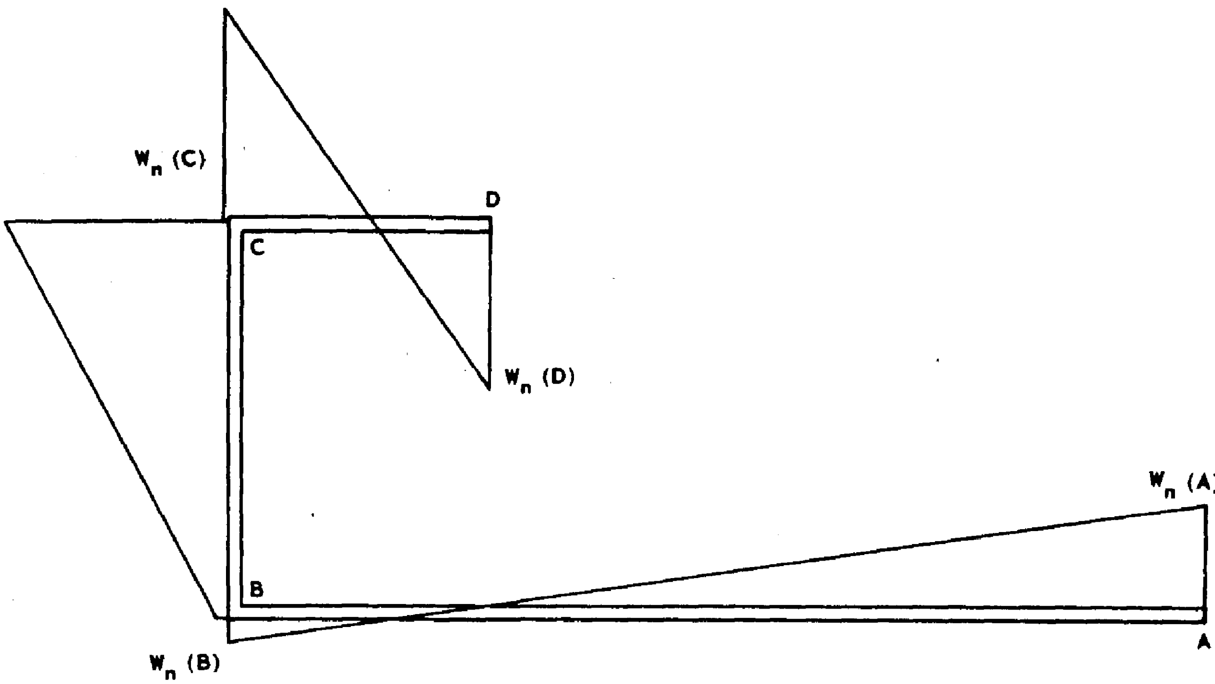


Figure B8.4.4-2. Distribution of $W_n(s)$

Warping Deformations (measured from mean displacement plane):

$$W(s) = \phi' \, W_n(s) \qquad \text{(Section B8.4.2-III)}$$

$$W(s) = \frac{M}{GK} W_n(s)$$

$$W(A) = \frac{100\ (0.388)}{3 \times 10^6 \times 3.285 \times 10^{-3}}$$

$W(A) = 0.0039$ in.

$W(B) = -0.0003$ in.

$W(C) = -0.0079$ in.

$W(D) = 0.0065$ in.

Evaluate K:

$$K = 1/3\ bt^3$$

$$K = 1/3(3.38 + 1.38 + 0.94)\ (0.12)^3$$

$$K = 3.285 \times 10^{-3}\ in.^4$$

Maximum angle of twist:

$$\phi\ (max) = \frac{M_t L}{GK} \qquad \text{(Section B8.4.2-I)}$$

$$\phi\ (max) = \frac{100 \times 41}{3 \times 10^6 \times 3.285 \times 10^{-3}}$$

$$\phi\ (max) = 0.416\ radian\ .$$

Torsional Shear Stress:

$$\tau_t = \frac{M_t}{S_t(s)} \qquad \text{(Section B8.4.2-II)}$$

where

$$S_t(s) = \frac{K}{t}$$

$$\tau_t = \frac{100 \times 0.12}{3.285 \times 10^{-3}} = 3655\ psi\ .$$

B8.4.3 RESTRAINED TORSION

I. ANGLE OF TWIST AND DERIVATIVES

It was shown that for unrestrained torsion, the torsional moment resisted by the section is $M_1 = GK\,\phi'$ (Section B8.4.2-I).

Longitudinal bending occurs when a section is restrained from free warping. This bending is accompanied by shear stresses in the plane of the cross section, and these stresses resist the external applied torsional moment according to the following relationship:

$$M_2 = -\,E\,\Gamma\,\phi'''$$

where

M_2 = resisting moment caused by restrained warping of the cross section, in.-lb

E = modulus of elasticity, psi

Γ = warping constant for the cross section (Section B8.4.1-IVB), in.^{6}

ϕ''' = third derivative of the angle of rotation with respect to x.

Therefore, the total torsional moment resisted by the section is the sum of M_1 and M_2. The first of these is always present; the second depends on the resistance to warping. Denoting the total torsional resisting moment by M, the following expression is obtained.

$$M = M_1 + M_2 = GK\phi' - E\Gamma\phi'''$$

or

$$\frac{1}{a^2}\,\phi' - \phi''' = \frac{M}{E\Gamma}$$

where

$$a^2 = \frac{E\Gamma}{GK} \; .$$

The solution of this equation depends upon the distribution of applied torque (M) and the boundary or end restraints of the member. Numerical evaluation of this equation for ϕ, ϕ', ϕ'', and ϕ''' is obtained from a computer program in the Astronautics Computer Utilization Handbook for many loading and end conditions.

It is necessary to evaluate the foregoing expressions for the angle of twist and its derivatives before a complete picture of stress distribution and warping can be defined.

B8.4.3 RESTRAINED TORSION

II. STRESSES

A. Pure Torsional Shear Stress

The equation for torsional shear stress is the same as given in Section B8.4.2-II; however, now the angle of twist varies along the member and must be determined from the previous section.

Neglecting stress concentrations at reentrant corners, the pure torsional shear stress equation is

$$\tau_t = Gt\,\phi'.$$

This stress will be largest in the thickest element of the cross section. For distribution of this stress for common sections, see Figures B8.4.1-3, B8.4.1-4, and B8.4.1-5. This stress can be calculated by a computer program from the Astronautics Computer Utilization Handbook for many loading and end conditions.

B. Warping Shear Stress

When the cross section is restrained from warping freely along the entire length of the member, warping shear stresses are induced. These stresses are essentially uniform over the thickness (t), but the magnitude varies at different locations of the cross section (Figs. B8.4.1-3, B8.4.1-4, and B8.4.1-5). These stresses are determined from the equation:

$$\tau_{ws} = -\frac{ES_{ws}}{t}\,\phi'''$$

where

τ_{ws} = warping shear stress at point s, psi

E = modulus of elasticity, psi

S_{ws} = warping statical moment at point s (Section B8.4.1-IVD), $in.^4$

t = thickness of the element, in.

ϕ''' = third derivative of the angle of twist with respect to x, distance measured along the length of the member.

This stress can be calculated by a computer program from the Astronautic Computer Utilization Handbook for many loading and end conditions.

C. Warping Normal Stress

Warping normal stresses are caused when the cross section is restrained from warping freely along the entire length of the member. These stresses act perpendicular to the surface of the cross section and are constant across the thickness of an element but vary in magnitude along the length of the element. The magnitude of these stresses is determined by the equation:

$$\sigma_{ws} = E\,W_{ns}\,\phi''$$

where

σ_{ws} = warping normal stress at point s, psi

E = modulus of elasticity, psi

W_{ns} = normalized warping function at point s (Sec.B8.4.1-IVC), $in.^2$

ϕ'' = record derivative of the angle of twist with respect to x, distance measured along the length of the member.

This stress can be calculated by a computer program from the Astronautic Computer Utilization Handbook for many loading and end conditions.

B8.4.3 RESTRAINED TORSION

III. WARPING DEFORMATIONS

Warping deformations can be calculated by using the same equation that was given in Section B8.4.2-III, except that now ϕ' will vary along the length of the member. The expression for ϕ' can be obtained from Section B8.4.3-I or values can be obtained from a program given in the Astronautic Computer Utilization Handbook. It should be noted that the warping normal stresses are proportional to corresponding warping displacements; hence, by knowing the warping displacements, a picture of distribution of the warping stresses is evident.

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