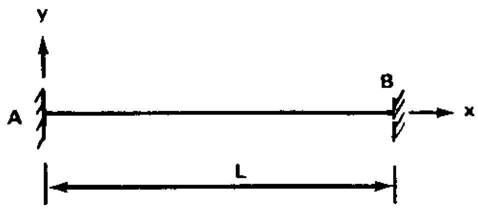
II. Fixed-Fixed Beam.



A. Boundary Conditions:

$$v = \frac{dv}{dx} = 0, \quad @ \quad x = 0, L \ ,$$

$$v(x) = - \int_0^x \int_0^{x_2} \left[\frac{M_{T_z}(x_1) + M_{0_z} + x_1 V_{0_z}}{EI_z(x_1)} \right] dx_1 \, dx_2 \qquad ,$$

$$v\left(\frac{x}{L}\right) = - \frac{\alpha F_1}{I_{z_0}} L^2 \left(I_{1x} + F_2 I_{2x} + F_3 I_{3x} \right) \qquad ,$$

where

$$I_{2x} = \int_0^x \int_0^{x_2} \frac{dx}{h(x_1)\ g^3(x_1)} \ dx_2 \quad , \quad I_2 = \int_0^1 \int_0^{x_2} \frac{dx}{h(x_1)\ g^3(x_1)} \ dx_2 \qquad ,$$

$$I_{3x} = \int_0^x \int_0^{x_2} \frac{x\ dx\ dx_2}{h(x_1)\ g^3(x_1)} \quad , \quad I_3 = \int_0^1 \int_0^{x_2} \frac{x\ dx}{h(x_1)\ g^3(x_1)} \ dx_2 \qquad ,$$

$$F_2 = \frac{I_3 \int_0^1 \frac{f(x_1)}{g^2(x_1)} - I_1 \int_0^1 \frac{x_1\ dx_1}{h(x_1)\ g^3(x_1)}}{I_2 \int_0^1 \frac{x_1\ dx_1}{h(x_1)\ g^3(x_1)} - I_3 \int_0^1 \frac{dx_1}{h(x_1)\ g^3(x_1)}} \qquad ,$$

$$F_3 = \frac{I_1 \int_0^1 \frac{dx_1}{h(x_1)\ g^3(x_1)} - I_2 \int_0^1 \frac{f(x_1)}{g^2(x_1)}\ dx_1}{I_2 \int_0^1 \frac{x_1\ dx_1}{h(x_1)\ g^3(x_1)} - I_3 \int_0^1 \frac{dx_1}{h(x_1)\ g^3(x_1)}} \quad .$$

(Table 3.0-4 gives values for F_2 and F_3)

$$M_{0_z} = \alpha\, E\, F_1\, F_2 \quad ,$$

$$V_{0_z} = \frac{\alpha}{L}\ E\, F_1\, F_3 \quad ,$$

$$M_z(x) = M_{0_z} + x\, V_{0_z} \quad ,$$

$$u_{av}(x) = \frac{\alpha\, F_0\, L}{A_0} \int_0^{x_1} f(x_1)\ dx_1 \quad .$$

If the end B is restrained against longitudinal motion, then

$$u_{av}(x) = 0 \quad ,$$

$$\sigma_{xx} = -\alpha\, ET + \left(\frac{M_{T_y} + M_y}{I_y}\right) z + \left(\frac{M_{T_z} + M_z}{I_z}\right) y \quad ,$$

$v(x)$, M_0, V_0 are same as above.

<u>CASE a</u>: $EI_z(x) = \text{constant}$, and $f(x_1) = \left(\frac{x}{L}\right)^q$.

$$v\left(\frac{x}{L}\right) = \frac{\alpha\, F_1\, x^2}{I_z\ (q+1)\ (q+2)} \left[\frac{x^q}{L^q} + q\left(1 - \frac{x}{L}\right) - 1\right] \quad ,$$

$$M_{0_z} = \frac{2\,\alpha\, E\, F_1 (q-1)}{(q+1)\ (q+2)} \quad ,$$

TABLE 3.0-4. VALUES OF CONSTANTS F_2, F_3, AND F_4

H = 0				H = 0.5				H = 1.0			
q = 0 (Uniform Temperature Distribution)											
G	F_2	F_3	F_4	G	F_2	F_3	F_4	G	F_2	F_3	F_4
0	-1.000	0.000	-1.502	0	-1.000	-0.500	-1.678	0	-1.000	-1.000	-1.837
0.1	-1.000	-0.100	-1.514	0.1	-0.993	-0.648	-1.684	0.1	-0.986	-1.191	-1.837
0.2	-1.000	-0.200	-1.525	0.2	-0.986	-0.794	-1.689	0.2	-0.974	-1.384	-1.838
0.3	-1.000	-0.300	-1.535	0.3	-0.981	-0.938	-1.694	0.3	-0.961	-1.571	-1.838
0.4	-1.000	-0.400	-1.544	0.4	-0.975	-1.081	-1.699	0.4	-0.957	-1.755	-1.839
0.5	-1.000	-0.500	-1.553	0.5	-0.970	-1.223	-1.704	0.5	-0.945	-1.937	-1.839
0.6	-1.000	-0.600	-1.561	0.6	-0.966	-1.364	-1.708	0.6	-0.937	-2.116	-1.840
0.7	-1.000	-0.700	-1.569	0.7	-0.962	-1.504	-1.712	0.7	-0.929	-2.291	-1.841
0.8	-1.000	-0.800	-1.577	0.8	-0.958	-1.642	-1.716	0.8	-0.922	-2.470	-1.842
0.9	-1.000	-0.900	-1.584	0.9	-0.955	-1.780	-1.719	0.9	-0.916	-2.644	-1.842
1.0	-1.000	-1.000	-1.590	1.0	-0.952	-1.918	-1.723	1.0	-0.910	-2.816	-1.843
q = 1 (Linear Temperature Distribution)											
G	F_2	F_3	F_4	G	F_2	F_3	F_4	G	F_2	F_3	F_4
0	0.000	-1.000	-0.502	0	0.077	-1.487	-0.561	0	0.114	-1.951	-0.614
0.1	0.016	-1.098	-0.490	0.1	0.095	-1.614	-0.545	0.1	0.165	-2.109	-0.586
0.2	0.030	-1.193	-0.479	0.2	0.111	-1.736	-0.530	0.2	0.183	-2.258	-0.572
0.3	0.042	-1.284	-0.469	0.3	0.126	-1.851	-0.517	0.3	0.199	-2.400	-0.561
0.4	0.054	-1.373	-0.459	0.4	0.139	-1.966	-0.505	0.4	0.213	-2.537	-0.546
0.5	0.064	-1.460	-0.450	0.5	0.151	-2.078	-0.494	0.5	0.226	-2.670	-0.533
0.6	0.074	-1.544	-0.442	0.6	0.161	-2.186	-0.483	0.6	0.237	-2.798	-0.520
0.7	0.083	-1.626	-0.434	0.7	0.171	-2.290	-0.473	0.7	0.248	-2.923	-0.509
0.8	0.091	-1.707	-0.426	0.8	0.180	-2.391	-0.464	0.8	0.257	-3.044	-0.498
0.9	0.099	-1.785	-0.419	0.9	0.188	-2.491	-0.455	0.9	0.266	-3.162	-0.488
1.0	0.106	-1.863	-0.413	1.0	0.196	-2.588	-0.447	1.0	0.274	-3.277	-0.478
q = 2 (Quadratic Temperature Distribution)											
G	F_2	F_3	F_4	G	F_2	F_3	F_4	G	F_2	F_3	F_4
0	0.167	-1.000	-0.252	0	0.243	-1.403	-0.281	0	0.310	-1.788	-0.309
0.1	0.176	-1.068	-0.241	0.1	0.253	-1.490	-0.268	0.1	0.321	-1.891	-0.292
0.2	0.184	-1.133	-0.231	0.2	0.261	-1.571	-0.256	0.2	0.329	-1.989	-0.279
0.3	0.191	-1.195	-0.222	0.3	0.268	-1.649	-0.246	0.3	0.336	-2.081	-0.267
0.4	0.197	-1.254	-0.215	0.4	0.274	-1.723	-0.236	0.4	0.342	-2.169	-0.256
0.5	0.202	-1.311	-0.207	0.5	0.279	-1.794	-0.227	0.5	0.347	-2.254	-0.246
0.6	0.206	-1.366	-0.201	0.6	0.283	-1.863	-0.219	0.6	0.351	-2.334	-0.236
0.7	0.210	-1.419	-0.194	0.7	0.287	-1.929	-0.212	0.7	0.354	-2.412	-0.228
0.8	0.214	-1.470	-0.189	0.8	0.290	-1.992	-0.205	0.8	0.357	-2.487	-0.220
0.9	0.217	-1.520	-0.183	0.9	0.293	-2.054	-0.199	0.9	0.359	-2.559	-0.213
1.0	0.220	-1.569	-0.178	1.0	0.296	-2.114	-0.193	1.0	0.361	-2.630	-0.207
q = 3 (Cubic Temperature Distribution)											
G	F_2	F_3	F_4	G	F_2	F_3	F_4	G	F_2	F_3	F_4
0	0.200	-0.900	-0.151	0	0.269	-1.238	-0.165	0	0.329	-1.559	-0.185
0.1	0.204	-0.948	-0.143	0.1	0.273	-1.299	-0.152	0.1	0.331	-1.631	-0.171
0.2	0.208	-0.994	-0.136	0.2	0.276	-1.356	-0.150	0.2	0.335	-1.698	-0.164
0.3	0.211	-1.037	-0.129	0.3	0.278	-1.409	-0.143	0.3	0.337	-1.763	-0.156
0.4	0.213	-1.078	-0.123	0.4	0.280	-1.460	-0.136	0.4	0.338	-1.822	-0.147
0.5	0.215	-1.118	-0.118	0.5	0.281	-1.509	-0.129	0.5	0.339	-1.879	-0.140
0.6	0.216	-1.156	-0.113	0.6	0.282	-1.555	-0.124	0.6	0.339	-1.933	-0.133
0.7	0.217	-1.192	-0.109	0.7	0.282	-1.600	-0.119	0.7	0.338	-1.986	-0.128
0.8	0.218	-1.227	-0.105	0.8	0.282	-1.643	-0.114	0.8	0.338	-2.036	-0.122
0.9	0.219	-1.261	-0.101	0.9	0.282	-1.685	-0.110	0.9	0.337	-2.084	-0.118
1.0	0.219	-1.294	-0.098	1.0	0.282	-1.725	-0.106	1.0	0.336	-2.131	-0.113

Notes: $h(x_1) = 1 + H\left(\frac{x}{L}\right)$

$g(x_1) = 1 + G\left(\frac{x}{L}\right)$

$$V_{0_z} = - \frac{6\,\alpha\, E\, F_1\, q}{(q+1)\,(q+2)L} \quad ,$$

$$M_z = \frac{2\,\alpha\, E\, F_1}{(q+1)\,(q+2)} \left[\left(1 - 3\,\frac{x}{L}\right) q - 1\right] \quad .$$

CASE b:

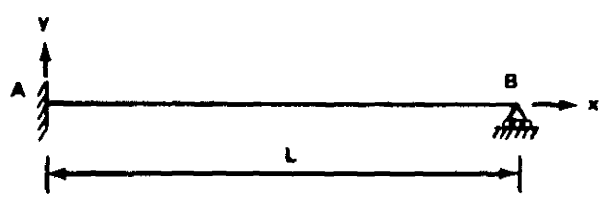
$$\frac{M_{T_z}(x_1)}{EI_z(x_1)} = \text{constant} \quad ,$$

$$v(x) = 0 \quad ,$$

$$M_{0_z} = -M_{T_z} \quad ,$$

$$V_{0_z} = 0 \qquad M_z = -M_{T_z} = \text{constant}.$$

III. Fixed-Hinged Beam.



A. Boundary Conditions:

$$v = \frac{dv}{dx} = 0, \quad @\ x = 0 \quad ,$$

$$v = \frac{d^2v}{dx^2} + \frac{M_{T_z}}{EI_z} = 0\,, \quad @\ x = L \quad ,$$

$$v\left(\frac{x}{L}\right) = -\frac{\alpha F_1 L^2}{I_{z_0}}\left[I_{x_1} + F_4\left(I_{x_2} - I_{x_3}\right)\right] \quad ,$$

where

$$F_4 = \frac{I_1}{(I_3 - I_2)} \quad \text{(refer to Table 3.0-4 for values of } F_4\text{)} ,$$

$$V_{0_z} = -\frac{\alpha E F_1 F_4}{L} \quad ,$$

$$M_{0_z} = \alpha E F_1 F_4 \quad ,$$

$$u_{av}(x) = \frac{\alpha F_0 L}{A_0} \int_0^{x_1} f(x_1)\, dx_1 \quad ,$$

$$\sigma_{xx} = -\alpha E T + \frac{P_T}{A} + \left(\frac{M_{T_z} + M_z}{I_z}\right) y + \left(\frac{M_{T_y} + M_y}{I_y}\right) z \quad .$$

If the end of B is hinged,

$$u_{av}(x) = 0 \quad ,$$

$$\sigma_{xx} = -\alpha E T + \left(\frac{M_{T_z} + M_z}{I_z}\right) y + \left(\frac{M_{T_y} + M_y}{I_y}\right) z \quad .$$

CASE a:

$$EI_z = \text{constant} \quad ,$$

$$f(x_1) = \left(\frac{x}{L}\right)^q \quad ,$$

$$v\left(\frac{x}{L}\right) = -\frac{\alpha F_1 L^2}{I_{0_z}} \left\{\left(\frac{x}{L}\right)^{q+2} - \frac{1}{2}\left[3\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3\right]\right\} \frac{1}{(q+1)(q+2)} ,$$

$$V_{0_z} = \frac{3\alpha E F_1}{(q+1)(q+2) L} ,$$

$$M_{0_z} = -\frac{3\alpha E F_1}{(q+1)(q+2)} ,$$

$$M_z = -\frac{3\alpha E F_1\left(1 - \frac{x}{L}\right)}{(q+1)(q+2)} .$$

CASE b:

$$\frac{M_{T_z}}{EI_z}(x) = \text{constant} ,$$

$$v(x) = \frac{\alpha F_1}{I_z} \frac{x^2}{4}\left(1 - \frac{x}{L}\right) ,$$

$$V_{0_z} = \frac{3\alpha E F_1}{L} ,$$

$$M_{0_z} = -\frac{3}{2}\alpha E F_1 ,$$

$$M_z = -\frac{3}{2}\left(1 - \frac{x}{L}\right)\alpha E F_1 .$$

IV. Deflection Plots.

From the previous cases the deflection expressions were found to be

A. Simply Supported Beam:

$$v(x) = \frac{F_1 L^2}{I_{0_z}}\left(\frac{x}{L} I_1 - I_{1x}\right)$$

B. Fixed-Fixed Beam:

$$v(x) = - \frac{F_1 L^2}{I_{0_Z}} \left(I_{1x} + F_2 I_{2x} + F_3 I_{3x} \right)$$

C. Fixed-Hinged Beam:

$$v(x) = - \frac{F_1 L^2}{I_{0_Z}} \left[I_{1x} + F_4 \left(I_{2x} - I_{3x} \right) \right]$$

These expressions have been plotted for parameter variations of beam width, depth, and degree of thermal gradient along the beam length. The plots are made in nondimensional form with the following designation:

$$SS = \left(\frac{x}{L} \; I_1 - I_{1x} \right) \quad ,$$

$$FS = - \left[I_{1x} + F_4 \left(I_{2x} - I_{3x} \right) \right] \quad ,$$

and

$$FF = - \left(I_{1x} + F_2 I_{2x} + F_3 I_{3x} \right) \quad .$$

Therefore the deflection, $v(x)$, for any case can be found by multiplying SS, FS, or FF by $\dfrac{F_1 L^2}{I_{0_Z}}$.

Figures 3.0-4 and 3.0-5 show the variation of FF and FS as a function of lengthwise temperature gradient. In the figure, $G = H = 0$ (constant cross section) and N equals the exponent of the thermal variation along the length of the beam; e.g., $N = 0$ means constant variation, $N = 1$ means linear variation, etc.

Figures 3.0-6 through 3.0-13 show the three deflection parameters SS, FS, and FF for variation of parameters G, H, and N, where H equals variation in width of beam along the length, G equals variation in depth at $x = 0$) (refer to Paragraph 3.0.3.1).

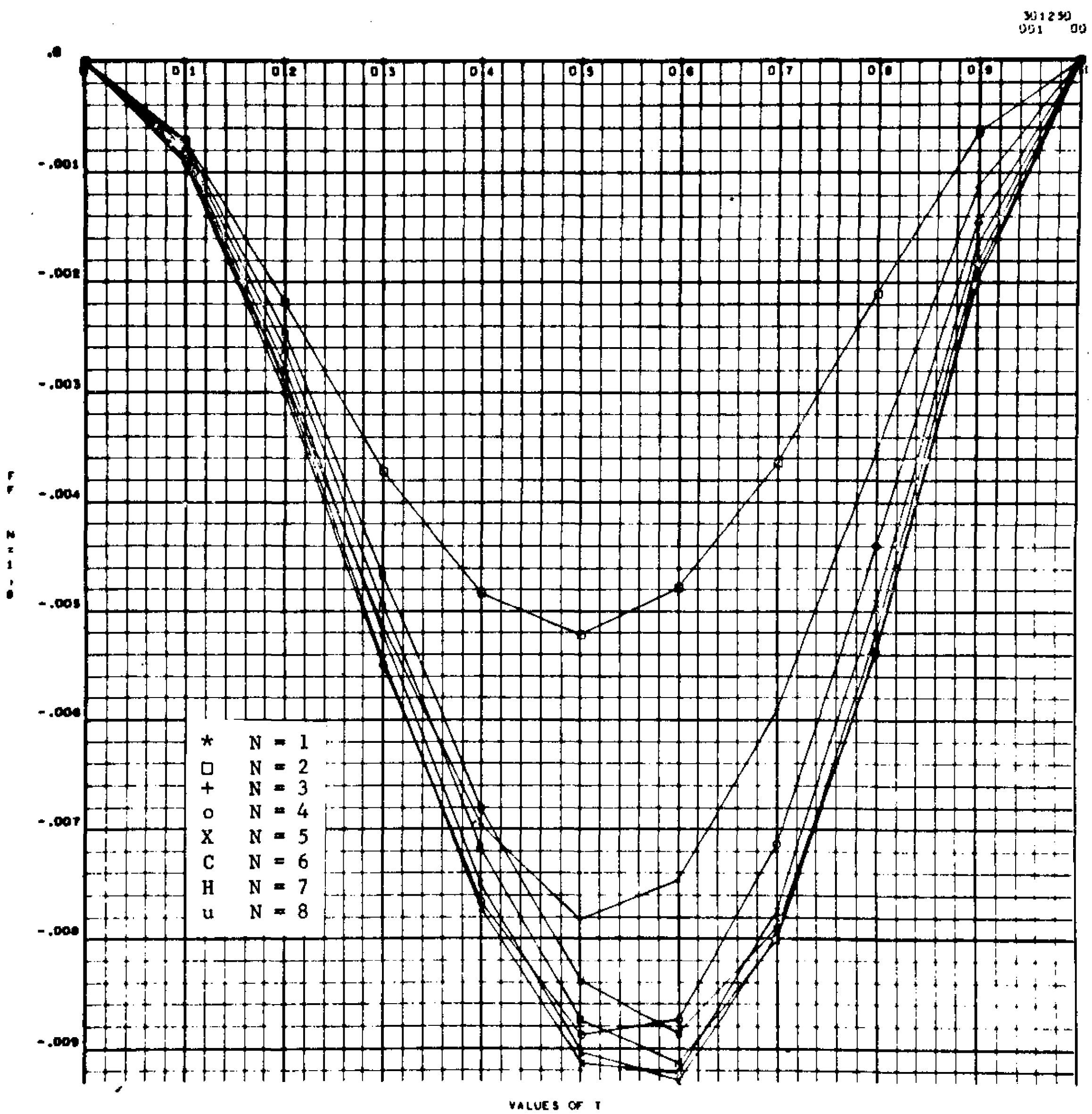


Figure 3.0-4. Deflection parameter FF versus distance along beam for variation of lengthwise temperature gradient.

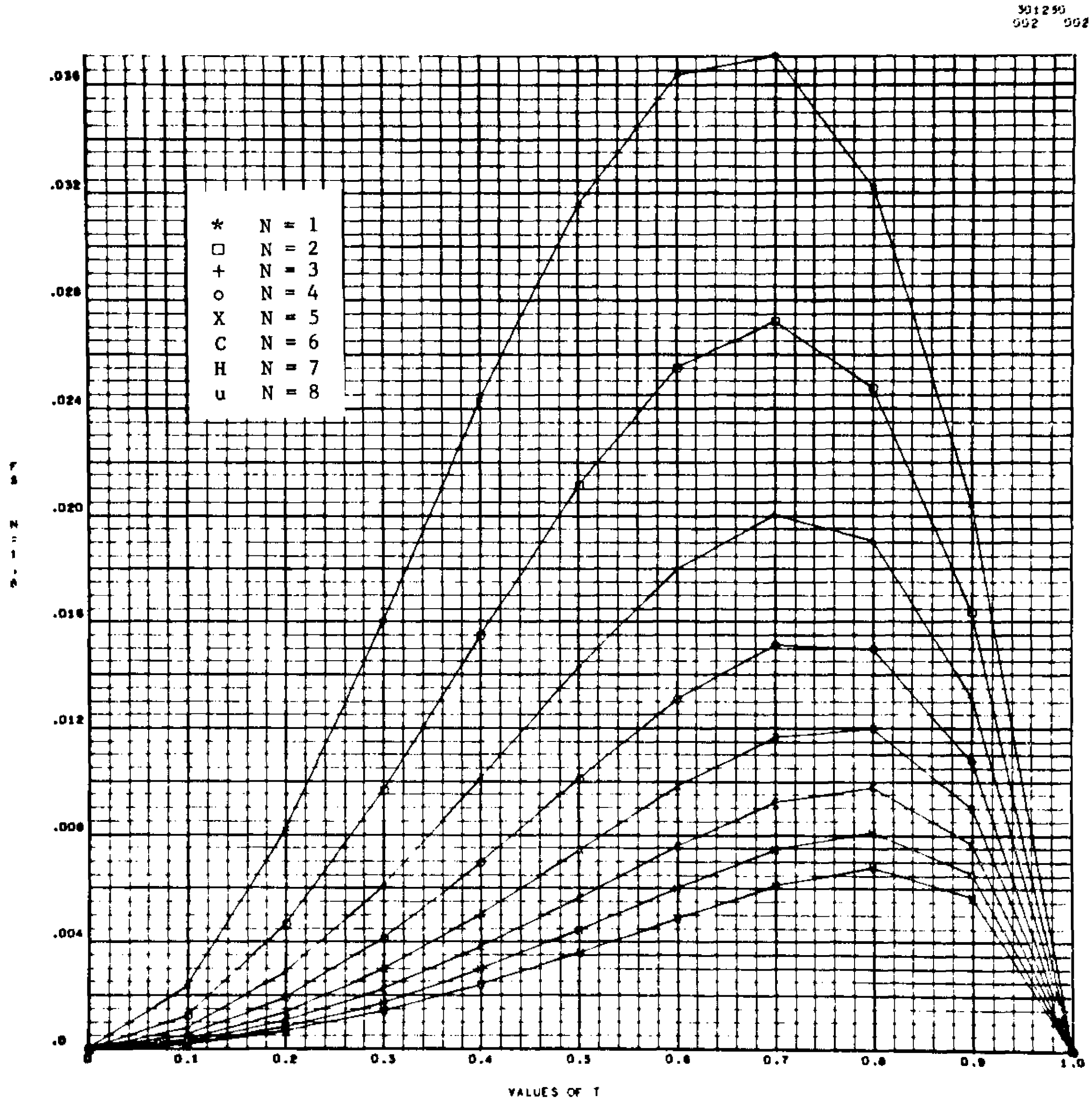


Figure 3.0-5. Deflection parameter FS versus distance along beam for variation of lengthwise temperature gradient.

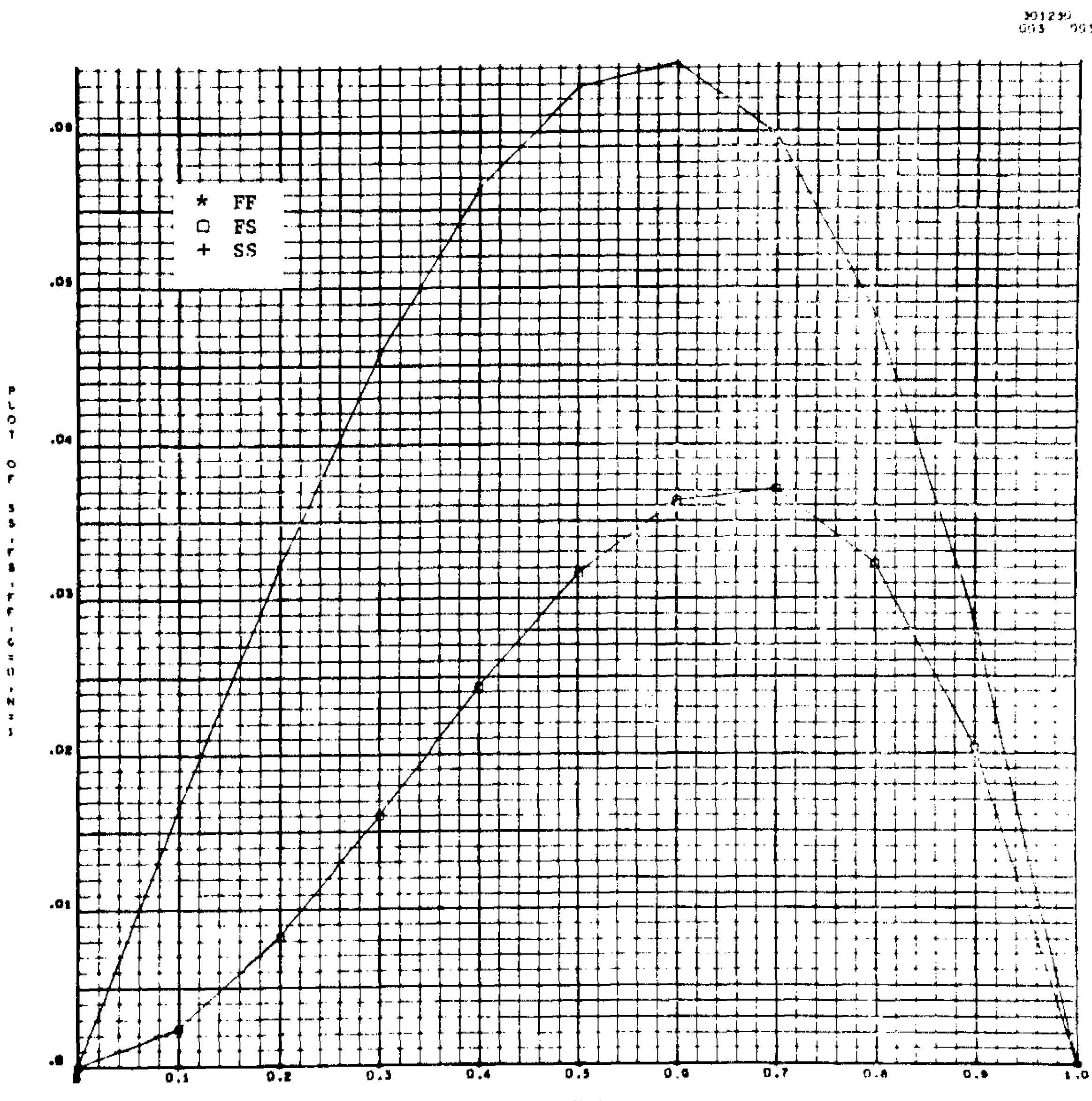


Figure 3.0-6. Values of SS, FS, FF versus values of T(x/L) for H = 0, G = 0, N = 1.

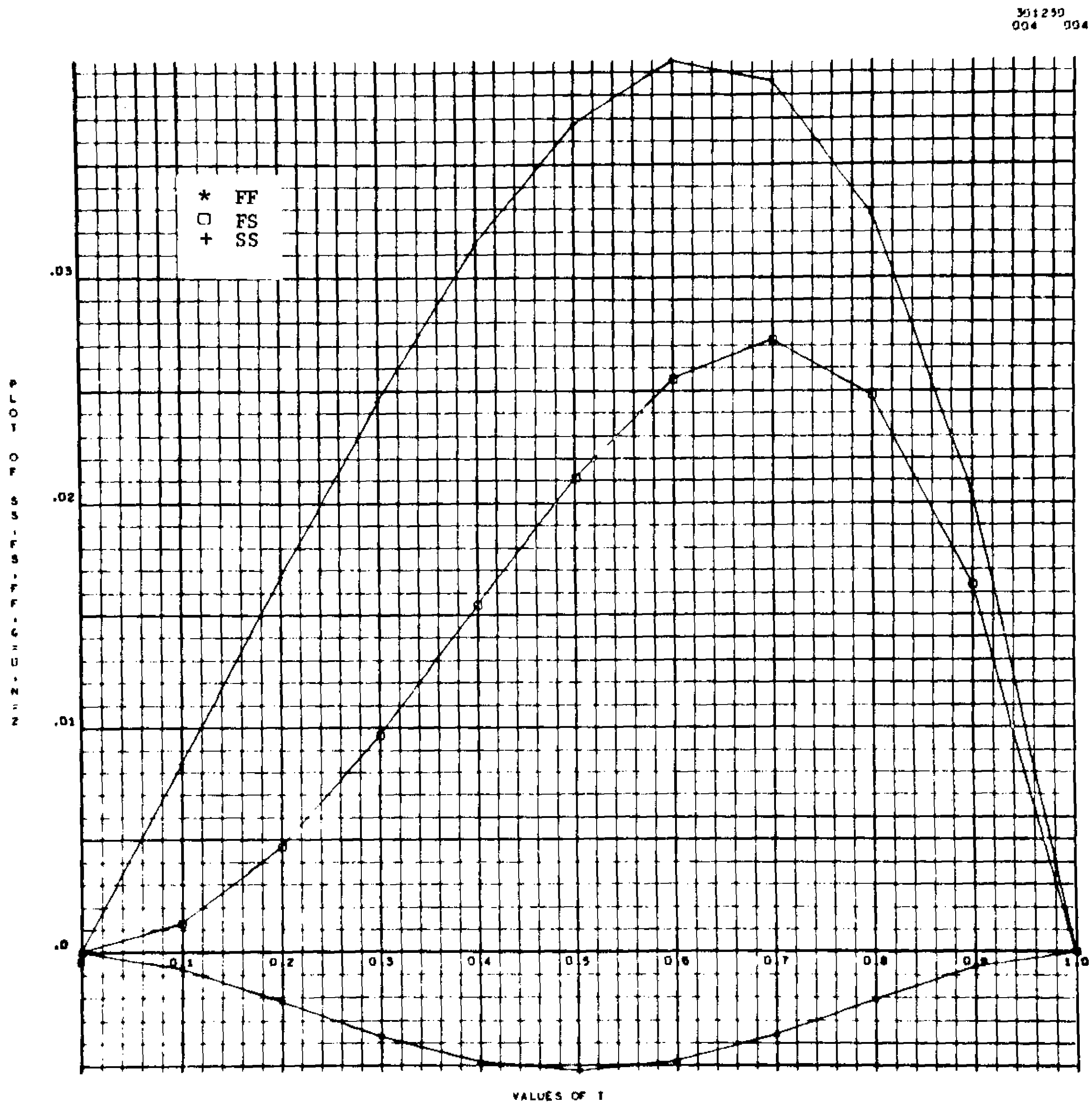


Figure 3.0-7. Values of SS, FS, FF versus values of T for H = 0, G = 0, N = 2.

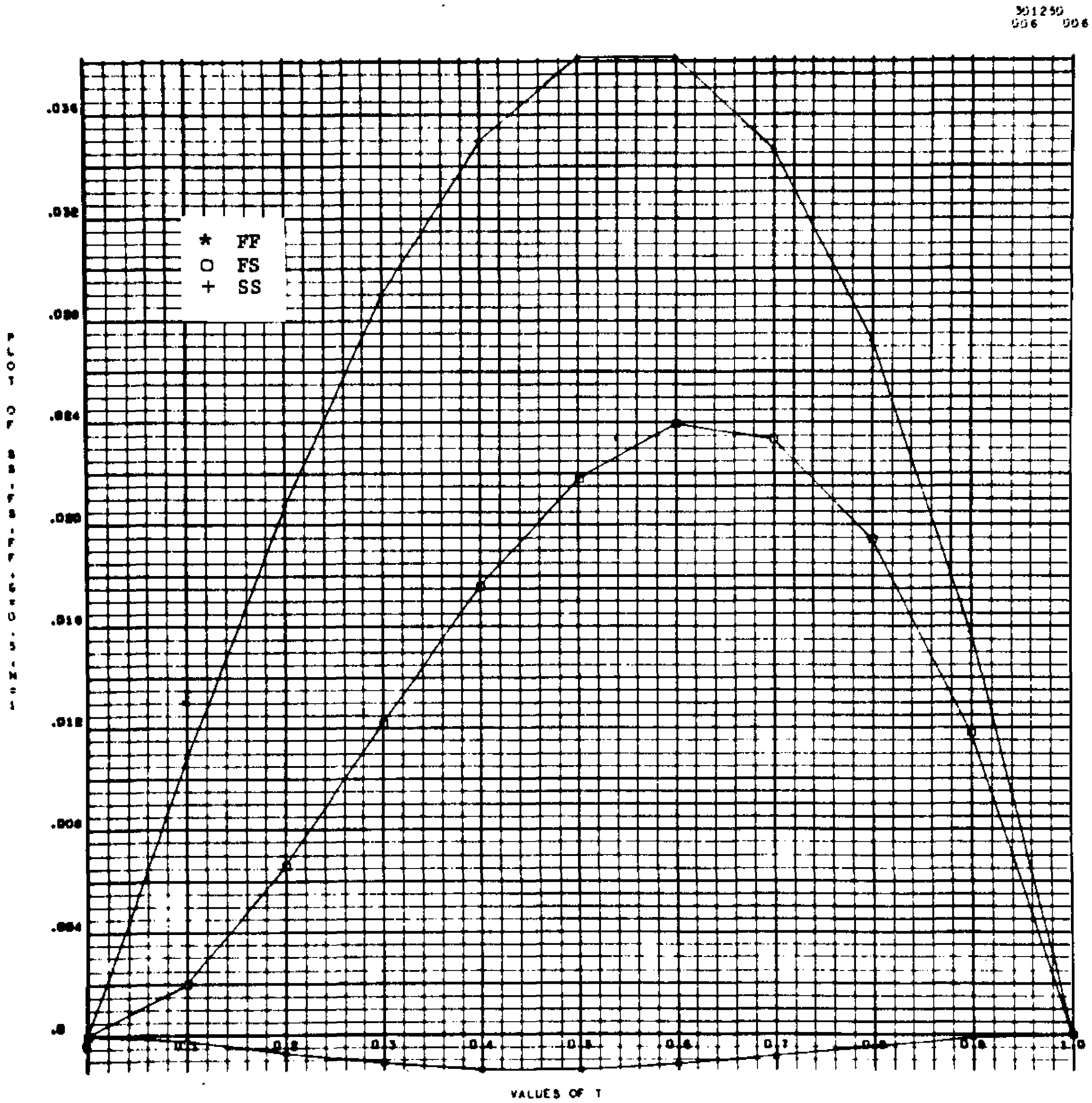


Figure 3.0-8. Values of SS, FS, FF versus values of T for H = 0, G = 0.5, N = 1.

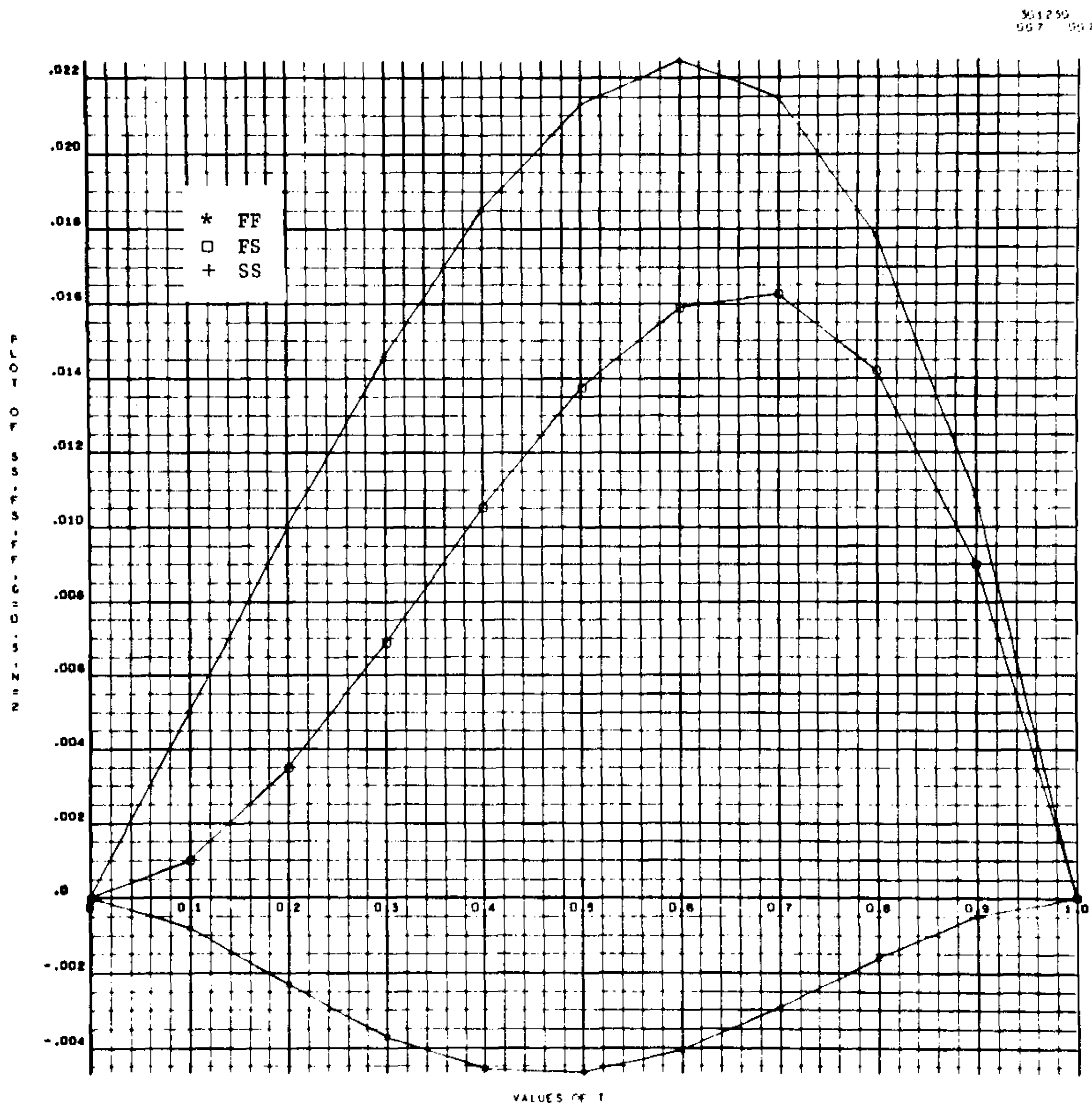


Figure 3.0-9. Values of SS, FS, FF versus values of T for H = 0, G = 0.5, N = 2.

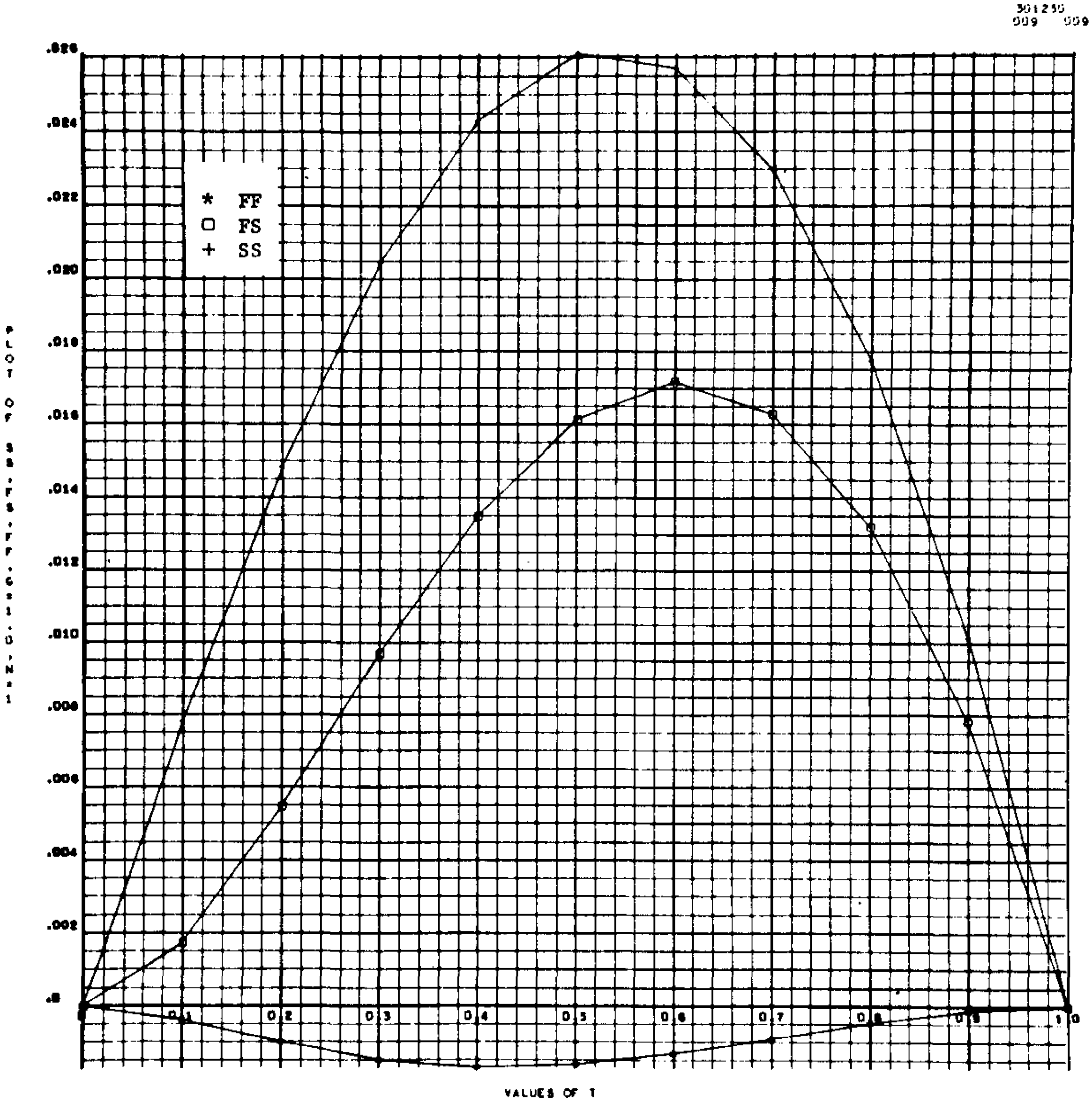


Figure 3.0-10. Values of SS, FS, FF versus values of T for H = 0, G = 1.0, N = 1.

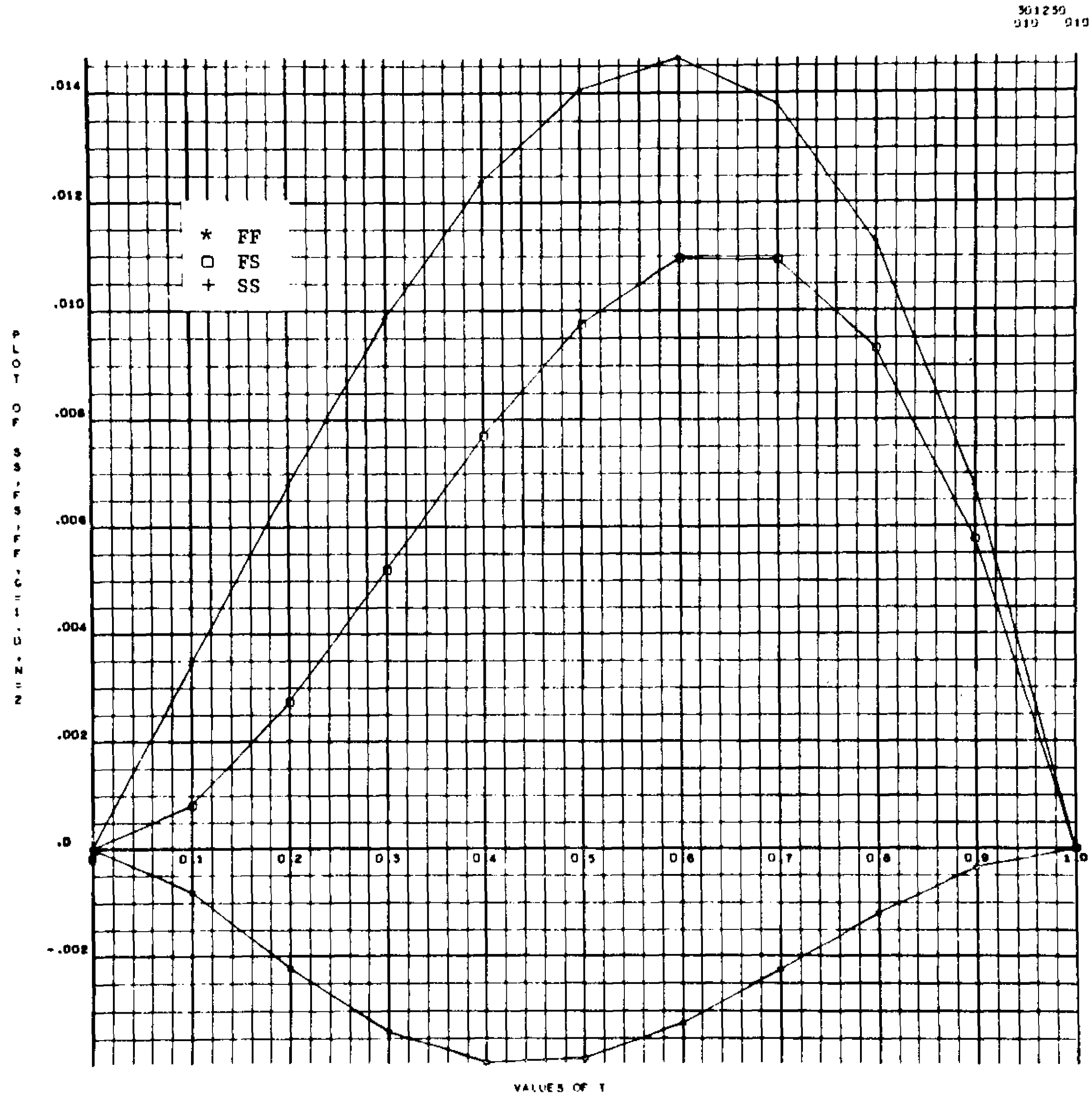


Figure 3.0-11. Values of SS, FS, FF versus values of T for H = 0, G = 1.0, N = 2.

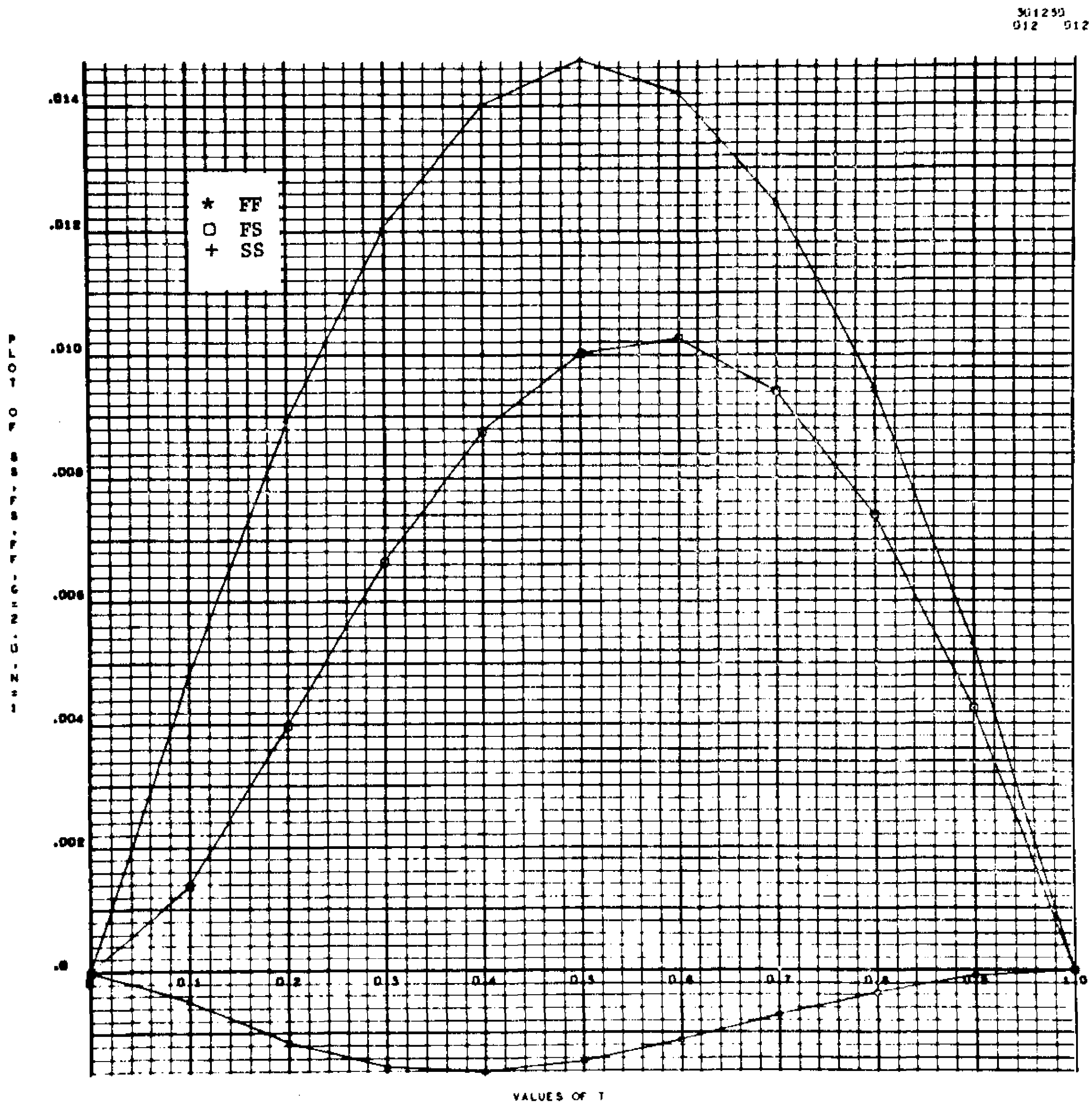


Figure 3.0-12. Values of SS, FS, FF versus values of T for H = 0, G = 2.0, N = 2.

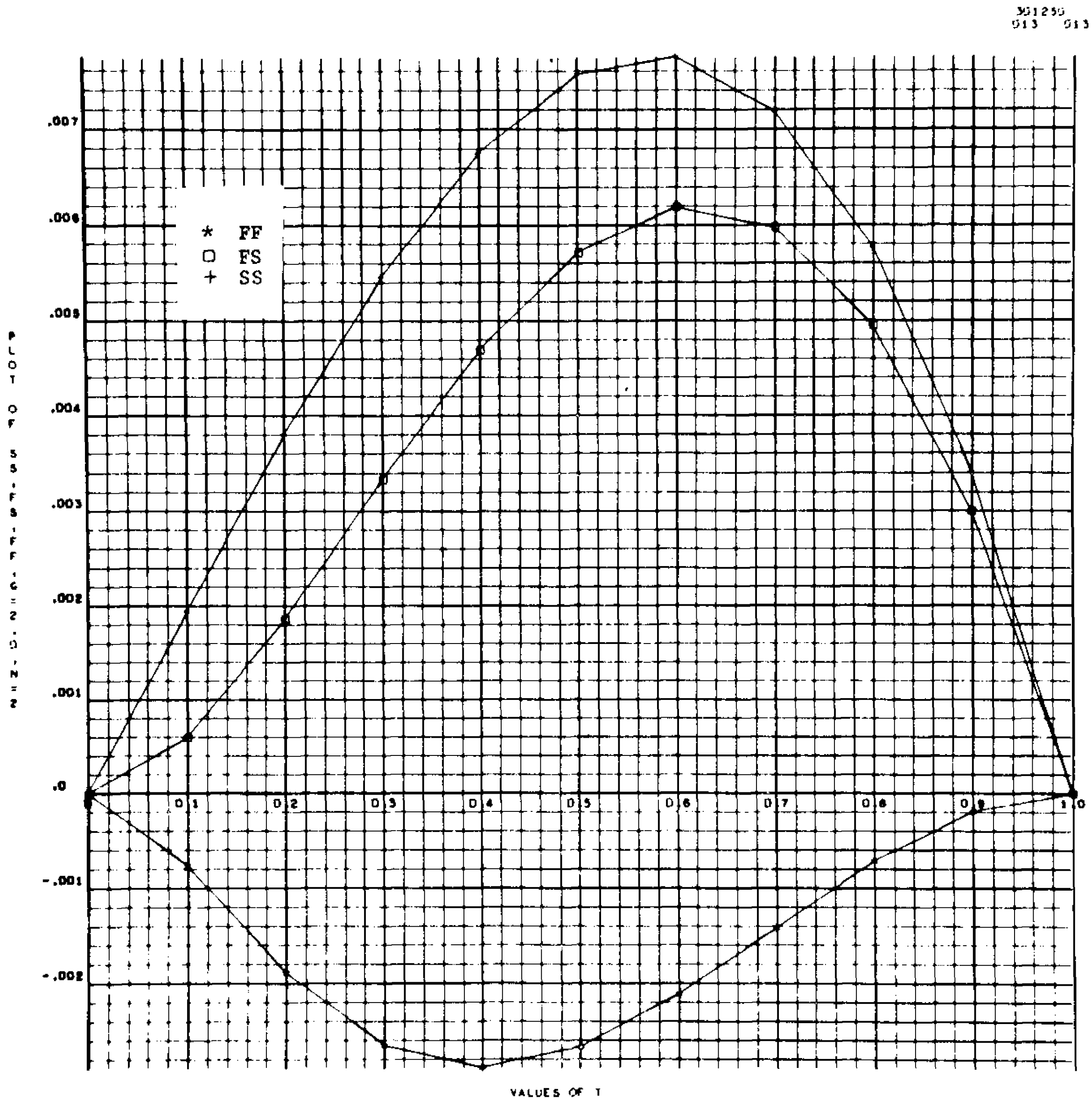


Figure 3.0-13. Values of SS, FS, FF versus values of T for H = 0, G = 2.0, N = 2.

3.0.2.3 Representation of Temperature Gradient by Polynomial.

A temperature profile obtained analytically or experimentally can be approximated by a polynomial of the form

$$T = \sum_{n=1}^{N} \sum_{m=1}^{M} V_{mn}\, y^{n} z^{m} + T_0 \quad .$$

Accuracy of this approximation increases with the increase in the number of terms of the polynomial; but, since the temperature distribution is to be integrated in the process of obtaining the stress distribution, the accuracy of the resulting distribution using a low-order polynomial improves. The total number of grid points required to determine the coefficients V_{mn} uniquely is equal to $(M+N)$.

The coefficients V_{mn} are obtained from the following relations.

$$\left\{\begin{matrix} V_{11} \\ V_{12} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ V_{1N} \\ V_{21} \\ V_{22} \\ \cdot \\ \cdot \\ V_{2N} \\ \cdot \\ \cdot \\ V_{MN} \end{matrix}\right\} = \begin{bmatrix} a_{1,11} & a_{1,12} & a_{1,13} & \cdots & a_{1,1N} & a_{1,21} & \cdots & a_{1,MN} \\ a_{2,11} & a_{2,12} & a_{2,13} & \cdots & a_{2,1N} & a_{2,21} & \cdots & a_{2,MN} \\ & & & & & & & \\ a_{N,11} & a_{N,12} & \cdots & & & & \cdots & a_{N,MN} \\ & & & & & & & \\ a_{(M+N),11} & a_{(M+N),12} & \cdots & & & & \cdots & a_{(M+N),MN} \end{bmatrix} \left\{\begin{matrix} T_1 - T_0 \\ T_2 - T_0 \\ \\ T_N - T_0 \\ \\ T_{(M+N)} - T_0 \end{matrix}\right\}$$

where $a_{i,mn} = y_i^{\,n} z_i^{\,m}$. The values (y_i, z_i) = coordinates of the ith grid point at temperature T_i and T_0 = temperature at the centroid of the cross section.

If the temperature difference $(T_i - T_0)$ is continuous and symmetric about the z-axis, then $V_{m1} = V_{m3} = V_{m5} = \ldots = 0$, and if it is continuous and antisymmetric $V_{m2} = V_{m4} = V_{m6} = \ldots\ 0$. Similarly, for temperature differences that are symmetric about the y-axis, $V_{1n} = V_{3n} = V_{5n} = \ldots\ 0$, and if it is antisymmetric about the y-axis, $V_{2n} = V_{4n} = V_{6n} = \ldots = 0$.

If the temperature varies only in one direction (e.g., for variation along the y-direction only) along either the y- or z-axis, the polynomial reduces to

$$T - T_0 = \sum_{n=1}^{N} V_n\, y^n \quad .$$

$$\begin{Bmatrix} V_1 \\ V_2 \\ \cdot \\ \cdot \\ \cdot \\ V_n \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{Bmatrix} T_1 - T_0 \\ T_2 - T_0 \\ \\ T_n - T_0 \end{Bmatrix} \quad .$$

$$a_{1n} = y_i^{\,n}$$

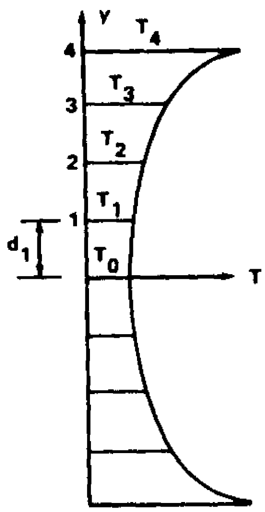
As before, if the temperature difference is continuous and symmetric, $V_1 = V_3 = V_5 = \ldots\ 0$; and for continuous antisymmetric temperature differences $V_2 = V_4 = V_6 = \ldots\ 0$.

If the grid points are equally spaced on the cross section, say at distance d_1, then the polynomial can be written in nondimensional form as

$$T_1 - T_0 = \sum_{n=1}^{N} V_n\, d_1^{\,n} \left(\frac{y}{d_1}\right)^n \quad .$$

EXAMPLES:

1. Continuous Symmetric Temperature



a. Nine Points:

$$T_i - T_0 = \sum_{n=2,4,6} (V_n d_1^n)\,(i)^n$$

$$\begin{Bmatrix} V_2 d_1^2 \\ V_4 d_1^4 \\ V_6 d_1^6 \\ V_8 d_1^8 \end{Bmatrix} = \begin{bmatrix} 1^2 & 1^4 & 1^6 & 1^8 \\ 2^2 & 2^4 & 2^6 & 2^8 \\ 3^2 & 3^4 & 3^6 & 3^8 \\ 4^2 & 4^4 & 4^6 & 4^8 \end{bmatrix}^{-1} \begin{Bmatrix} T_1 - T_0 \\ T_2 - T_0 \\ T_3 - T_0 \\ T_4 - T_0 \end{Bmatrix}$$

$$\begin{Bmatrix} V_2 d_1^2 \\ V_4 d_1^4 \\ V_6 d_1^6 \\ V_8 d_1^8 \end{Bmatrix} = \begin{bmatrix} 0.16000{+}01 & -0.20000{+}00 & 0.25397{-}01 & -0.17857{-}02 \\ -0.67778{+}00 & 0.23472{+}00 & -0.33333{-}01 & 0.24306{-}02 \\ 0.80556{-}01 & -0.36111{-}01 & 0.83333{-}02 & -0.69444{-}03 \\ -0.27778{-}02 & 0.13889{-}02 & -0.39683{-}03 & 0.49603{-}04 \end{bmatrix} \begin{Bmatrix} T_1 - T_0 \\ T_2 - T_0 \\ T_3 - T_0 \\ T_4 - T_0 \end{Bmatrix}$$

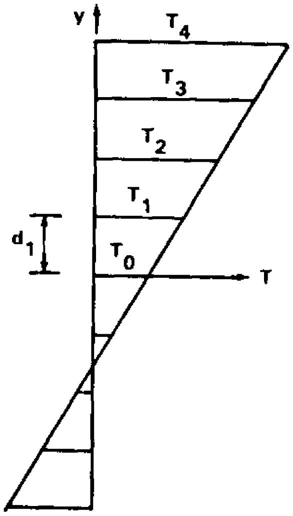
b. Seven Points:

$$\begin{Bmatrix} V_2 d_1^2 \\ V_4 d_1^4 \\ V_6 d_1^6 \end{Bmatrix} = \begin{bmatrix} 0.15000{+}01 & -0.15000{+}00 & 0.11111{-}01 \\ -0.54167{+}00 & 0.16667{+}00 & -0.13889{-}01 \\ 0.41667{-}01 & -0.16667{-}01 & 0.27778{-}02 \end{bmatrix} \begin{Bmatrix} T_1 - T_0 \\ T_2 - T_0 \\ T_3 - T_0 \end{Bmatrix}$$

c. Five Points:

$$\begin{Bmatrix} V_2 d_1^2 \\ V_4 d_1^4 \end{Bmatrix} = \begin{bmatrix} 1^2 & 1^4 \\ 2^2 & 2^4 \end{bmatrix}^{-1} \begin{Bmatrix} T_1 - T_0 \\ T_2 - T_0 \end{Bmatrix} = \begin{bmatrix} \frac{4}{3} & -\frac{1}{3} \\ -\frac{1}{12} & \frac{1}{12} \end{bmatrix} \begin{Bmatrix} T_1 - T_0 \\ T_2 - T_0 \end{Bmatrix}$$

2. Continuous Antisymmetric Temperature



a. Nine Points:

$$\begin{Bmatrix} V_1 d_1 \\ V_3 d_1^3 \\ V_5 d_1^5 \\ V_7 d_1^7 \end{Bmatrix} = \begin{bmatrix} 1 & 1^3 & 1^5 & 1^7 \\ 2 & 2^3 & 2^5 & 2^7 \\ 3 & 3^3 & 3^5 & 3^7 \\ 4 & 4^3 & 4^5 & 4^7 \end{bmatrix}^{-1} \begin{Bmatrix} T_1 - T_0 \\ T_2 - T_0 \\ T_3 - T_0 \\ T_4 - T_0 \end{Bmatrix}$$

$$\begin{Bmatrix} V_1 d_1 \\ V_3 d_1^3 \\ V_5 d_1^5 \\ V_7 d_1^7 \end{Bmatrix} = \begin{bmatrix} 0.15003{+}01 & -0.30031{+}00 & 0.33466{-}01 & -0.22126{-}04 \\ -0.54209{+}00 & 0.33375{+}00 & -0.41847{-}01 & 0.30116{-}04 \\ 0.41787{-}01 & -0.33454{-}01 & 0.83850{-}02 & -0.86045{-}05 \\ -0.86045{-}05 & 0.86045{-}05 & -0.36876{-}05 & 0.61461{-}06 \end{bmatrix} \begin{Bmatrix} T_1 - T_0 \\ T_2 - T_0 \\ T_3 - T_0 \\ T_4 - T_0 \end{Bmatrix}$$

b. Seven Points:

$$\begin{Bmatrix} V_1 d_1 \\ V_3 d_1^3 \\ V_5 d_1^5 \end{Bmatrix} = \begin{bmatrix} 0.15000{+}01 & -0.30000{+}00 & 0.33333{-}01 \\ -0.54167{+}00 & 0.33333{+}00 & -0.41667{-}01 \\ 0.41667{-}01 & -0.33333{-}01 & 0.83333{-}02 \end{bmatrix} \begin{Bmatrix} T_1 - T_0 \\ T_2 - T_0 \\ T_3 - T_0 \end{Bmatrix}$$

c. Five Points:

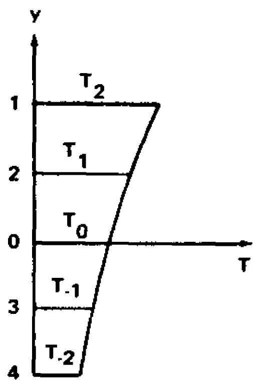
$$\begin{Bmatrix} V_1 d_1 \\ V_3 d_1^3 \end{Bmatrix} = \begin{bmatrix} 1 & 1^3 \\ 2 & 2^3 \end{bmatrix}^{-1} \begin{Bmatrix} T_1 - T_0 \\ T_2 - T_0 \end{Bmatrix} = \begin{bmatrix} \frac{4}{3} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} \end{bmatrix} \begin{Bmatrix} T_1 - T_0 \\ T_2 - T_0 \end{Bmatrix}$$

d. Three Points:

$$V_1 d_1 = T_1 - T_0$$

3. Arbitrary Temperature Distribution

a. Five Points:



$$\begin{Bmatrix} V_1\, d_1 \\ V_2\, d_1^2 \\ V_3\, d_1^3 \\ V_4\, d_1^4 \end{Bmatrix} = \begin{bmatrix} 2^1 & 2^2 & 2^3 & 2^4 \\ 1^1 & 1^2 & 1^3 & 1^4 \\ (-1)^1 & (-1)^2 & (-1)^3 & (-1)^4 \\ (-2)^1 & (-2)^2 & (-2)^3 & (-2)^4 \end{bmatrix}^{-1} \begin{Bmatrix} T_2 - T_0 \\ T_1 - T_0 \\ T_{-1} - T_0 \\ T_{-2} - T_0 \end{Bmatrix}$$

$$\begin{Bmatrix} V_1\, d_1 \\ V_2\, d_1^2 \\ V_3\, d_1^3 \\ V_4\, d_1^4 \end{Bmatrix} = \begin{bmatrix} -0.83333-01 & 0.66667+00 & -0.66667+00 & 0.83333-01 \\ -0.41667-01 & 0.66667+00 & 0.66667+00 & -0.41667-01 \\ 0.83333-01 & -0.16667+00 & 0.16667+00 & -0.83333-01 \\ 0.41667-01 & -0.16667+00 & -0.16667+00 & 0.41667-01 \end{bmatrix} \begin{Bmatrix} T_2 - T_0 \\ T_1 - T_0 \\ T_{-1} - T_0 \\ T_{-2} - T_0 \end{Bmatrix}$$

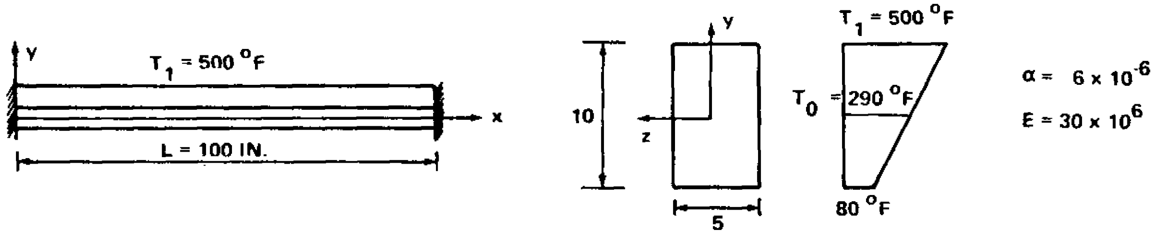
b. Three Points:

$$\begin{Bmatrix} V_1\, d_1 \\ V_2\, d_1^2 \end{Bmatrix} = \begin{bmatrix} 1^1 & 1^2 \\ (-1)^1 & (-1)^2 \end{bmatrix}^{-1} \begin{Bmatrix} T_1 - T_0 \\ T_{-1} - T_0 \end{Bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} T_1 - T_0 \\ T_{-1} - T_0 \end{Bmatrix}$$

Once the polynomial coefficients are determined, refer to Table 3.0-2 or 3.0-3 for constant F_1. Refer to Table 3.0-4 for the constants F_2, F_3, and F_4 to be used to find the fixed-end moments, reactions, and deflections of the beam.

I. Example Problem 1.

Given: The beam of rectangular cross section with a temperature distribution constant in the x-direction and varying linearly from 500°F on the top surface to 80°F on the bottom surface.



Find: Maximum stresses and maximum deflection $v(x)$ in center of beam.

Solution:

1. Find the polynomial expression representing the given temperature distribution in the y-direction. Refer to Paragraph 3.0.2.3, Continuous Antisymmetric Temperature, Three Points

$$v_1 \, d_1 = T_1 - T_0 \quad .$$

Therefore

$$v_1(5) = 500 - 290$$

$$v_1 = 42 \quad .$$

Therefore expression for $T = v_0 + v_1 \, y = 290 + 42 y$.

2. Find values of F_0, F_1, F_2, and F_3 for each term of polynomial. Refer to Table 3.0-2.

$v_0 = 290$ $\qquad F_0 = 290 \times b_0 d_0 = 290 \times 10(5) = 50 \times 290$ $\qquad F_1 = 0$

$v_1 = 42$ $\qquad F_0 = 0$

$$F_1 = \frac{(42)}{12} \, b_0 d_0^3 = \frac{42}{12} \, (5) \, (10)^3 = 416.67(42)$$

Refer to Table 3.0-4:

$(q = 0, \; H = 0, \; G = 0)$ $\qquad f(x) = 500 \left(\frac{x}{L}\right)^q$

$F_2 = -1.000$ $\qquad$ since constant in x-direction
$q = 0$

$F_3 = 0$ $\qquad$ constant depth $G = 0$
constant width $H = 0$

3. Refer to Paragraph 3.0.2.2 II (Case b)

$$M_{0_z} = \alpha E F_1 F_2$$

$$M_{0_z} = 6 \times 10^{-6} \times 30 \times 10^6 \times 416.67 \times 40 \times (-1.00)$$

$$= 3.15 \times 10^6$$

$$\sigma_{xx} = -\alpha E T + \frac{M_{0_z} y}{I_z}$$

Top Fiber: $\sigma_{xx} = -6 \times 10^6 \times 30 \times 10^6 \times 500 + \dfrac{(-3.15 \times 10^6) \times 5}{416.67}$

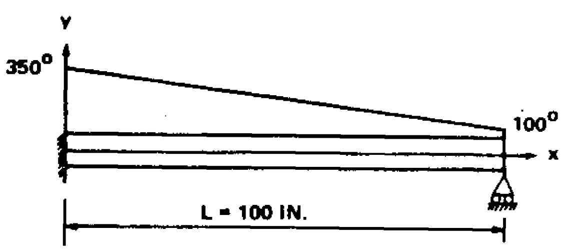
$$\sigma_{xx} = -127\ 800 \text{ psi}$$

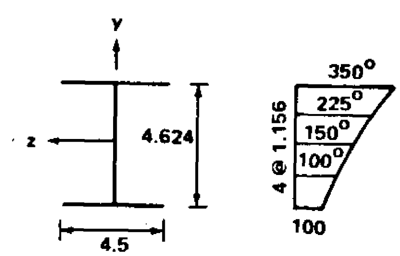
Deflection: $v(x) = 0$

Therefore the beam remains straight.

II. Example Problem 2.

Given: The I-beam shown below with a linear varying temperature in the x-direction and varying as shown in the y-direction.





Find: Stress σ_{xx} and deflection $v(x)$.

Solution:

1. Find the polynomial expression representing the given temperature distribution in the y-direction. Refer to Paragraph 3.0.2.3, Arbitrary Temperature Distribution, Five Points.

$$\begin{Bmatrix} V_1 d_1 \\ V_2 d_1^2 \\ V_3 d_1^3 \\ V_4 d_1^4 \end{Bmatrix} = \begin{bmatrix} -0.08333 & 0.6667 & -0.6667 & 0.08333 \\ -0.04167 & 0.6667 & 0.6667 & -0.04167 \\ 0.08333 & -0.16667 & 0.16667 & -0.08333 \\ 0.04167 & -0.1667 & -0.16667 & 0.04167 \end{bmatrix} \begin{Bmatrix} 350-150 \\ 225-150 \\ 100-150 \\ 100-150 \end{Bmatrix} = \begin{Bmatrix} 62.5 \\ 10.42 \\ 0 \\ 2.083 \end{Bmatrix}$$

$$T = 150 + \frac{62.5}{1.156}\, y + \frac{10.42}{(1.156)^2}\, y^2 + \frac{2.083}{(1.156)^4}\, y^4$$

$$T = 150 + 54.1y + 7.81y^2 + 1.168y^4$$

2. Refer to Tables 3.0-2 and 3.0-4 for values of F_0, F_1, F_2, F_3, and F_4 for each term of polynomial.

$n = 0$	$V_0 = 150$	$F_0 = 150(1.715) = 257.3$	$F_1 = 0$
$n = 1$	$V_1 = 54.1$	$F_0 = 54.1(0) = 0.0$	$F_1 = 54.1(2.774) = 150.0$
$n = 2$	$V_2 = 7.81$	$F_0 = 7.81(2.774) = 21.6$	$F_1 = 0$
$n = 4$	$V_4 = 1.168$	$F_0 = 1.168(6.745) = 7.9$	$F_1 = 0$
		$\Sigma F_0 = 286.8$	$F_1 = 150.0$

$$f\left(\frac{x}{L}\right) = \left(\frac{x}{L}\right)^q = 350 - 250\left(\frac{x}{L}\right)$$

$q = 0$	$F_2 = -1.000$	$F_3 = 0.0$	$F_4 = -1.502$
$q = 1$	$F_2 = 0.0$	$F_3 = -1.0$	$F_4 = -0.502$
	$F_2 = -1.000$	$F_3 = -1.0$	$F_4 = -2.004$

3. Refer to Section 3.0.2.2-III (Case b).

$$M_{0_z} = -3/2\,\alpha E F_1 = -3/2\,(180)(150) = -40,500$$

$$M_z = M_{0_z}(1 - x/L) \qquad I_z = 10.55 \text{ in}^4.$$

$$\sigma_{xx}\ (@x = 0) = -\ \alpha E T + \frac{M_{0_z}\, y}{I_z}$$

$$\sigma_{xx} = -180(350) - \frac{40\,000(2.312)}{10.55}$$

$$\sigma_{xx} = -71\,860 \text{ psi}$$

Deflection:

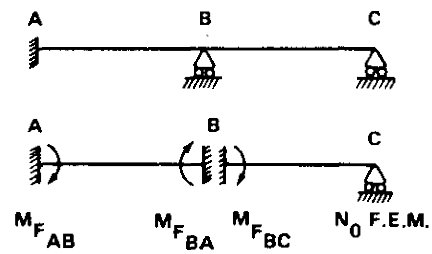
$$v(x) = \frac{\alpha F_1}{I_z}\, x^2/4(1 - x/L) = 21.35 \times 10^{-6}\, x^2(1 - x/L)$$

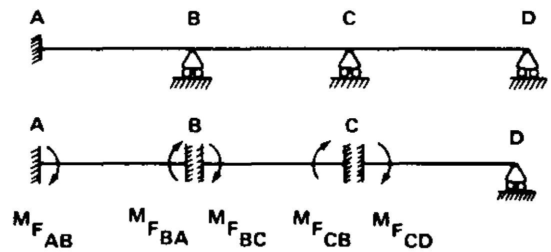
x	v(x)
0	0
10	$90 \times 21.35 \times 10^{-6}$
20	$320 \times$ "
40	960 "
50	1250 "
60	1440 "
80	1280 "
90	810 "
100	0

Refer also to Fig. 3.0-6 for the deflection.

3.0.3 Indeterminate Beams and Rigid Frames.

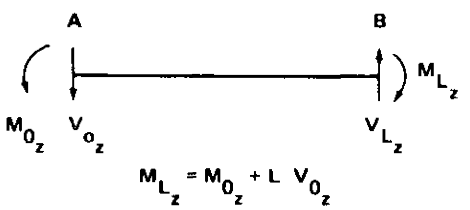
Continuous beams or frames can be analyzed by the method of moment distribution described in Section B5.0, Frames. In this method a continuous beam is fixed against rotation at all the intermediate supports over which the beam is continuous; and, in the case of a frame, all the rigid joints are also fixed against rotation and displacement. For each beam segment the moment required to keep the slope at each of the two supports unchanged is called a fixed-end moment. According to the convention of the method of moment distribution, the fixed-end moment, which is an externally applied moment, is positive if it is clockwise and negative if it is counterclockwise. This convention should not be confused with the convention of strength-of-materials for internal movement (refer to Paragraph 3.0.2). Fixed-end moment is designated by the subscript, F, and is abbreviated F.E.M.



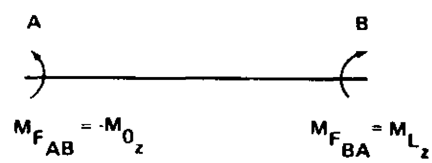


The magnitude of F.E.M. is obtained by analyzing the fixed-fixed beam or the fixed-hinged beam cases of Paragraph 3.0.2. In that paragraph, M_{0_Z} is positive when counterclockwise; so, to convert M_{0_Z} to F.E.M., one must multiply by -1.

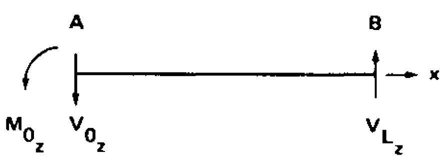
FIXED - FIXED BEAM



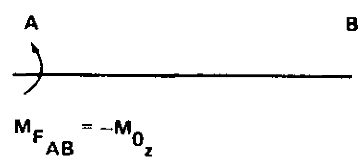
FIXED - END MOMENTS

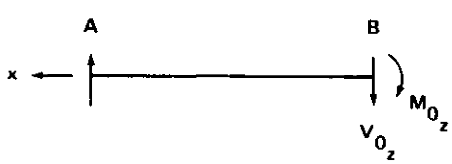


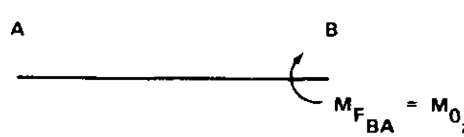
FIXED - HINGED BEAM



FIXED-END MOMENTS







With this information, a normal moment distribution technique can be performed to solve any continuous beam or frame.

3.0.4 Curved Beams.

For a free curved beam of arbitrary, constant cross section, the centerline of which is an arc of a circle lying in one of the centroidal principal planes under a temperature distribution $T(r,\theta)$, the following comments can be made.

1. For a small depth-to-radius ratio, straight-beam theory can be used instead of the curved-beam theory.

2. For linear temperature variation $\left(T = T_1 \frac{z}{h}\right)$, the results of curved-beam theory compare well with the solution obtained by exact thermoelasticity method; e.g., for a rectangular beam where the ratio of outside radius to inside radius is equal to three, the maximum stress obtained by the curved-beam theory differs from the maximum stress obtained by exact solution by only 2.4 percent. Straight-beam theory, however, gives $\sigma_{\theta\theta} = 0$, which is considerably erroneous.

3. For quadratic temperature distribution $T = T_2 \frac{z^2}{h^2}$, the difference in maximum stress obtained by curved-beam theory and exact solution is only 4.9 percent for a rectangular section where the ratio of outside to inside radius is equal to 3.0.

For solutions other than those mentioned above, see Ref. 1.

3.0.5 Rings.

A ring may be regarded as a closed, thin, curved beam, and can therefore be studied by the methods given in 3.0.4 and in Ref. 1.

3.0.6 Trusses.

3.0.6.1 Statically Determinate.

Every member of a pin-jointed determinate truss is not constrained to elongate because of thermal loading. As a result, the net axial force in each member is zero. Therefore, members of a pin-jointed determinate truss are regarded as simply supported beams under thermal loading and should be analyzed accordingly.

Deflections of joints of a truss can be obtained by the conventional "dummy-load" method described in Section B4.2.2.